The L^2-Alexander torsion of Seifert fiber spaces

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Einleitung

Motivation

Eine der fundamentalsten topologischen Invarianten ist die Euler-Charakteristik χ. Sie geht auf Leonhardt Euler zurück, der 1758 bewies, dass die Anzahl der Ecken plus die Anzahl der Flächen minus die Anzahl der Kanten eines konvexen Polyeders immer zwei beträgt. Seitdem gibt es eine rapide Entwicklung im Gebiet der Topologie, wobei die Euler-Charakteristik immer eine zentrale Rolle einnimmt. Sie ist zum Beispiel eine vollständige Invariante für geschlossene orientierbare zwei-dimensionalen Mannigfaltigkeiten. Zu den wichtigsten Eigenschaften gehören, dass diese

- leicht zu berechnen,
- invariant unter stetigen Deformationen und
- multiplikativ unter endlichen Überlagerungen ist.

Die moderne Definition der Euler-Charakteristik $\chi(X)$ eines topologischen Raumes X ist die alternierende Summe der Bettizahlen $b_i(X)$, woraus die Invarianz unter stetigen Deformationen folgt. Die Multiplikativität unter endlichen Überlagerungen liefern die Bettizahlen nicht. Eine Möglichkeit, diesen Defekt zu umgehen, ist die Definition von L^2-Bettizahlen $b_i^{(2)}(X)$, deren alternierende Summe ebenfalls die Euler-Charakteristik ist. Aus dieser Idee heraus hat man eine ganze Reihe von L^2-Invarianten gefunden, die jeweils ein Analogon zu den klassichen Invarianten bilden, mit der Eigenschaft multiplikativ unter endlichen Überlagerungen zu sein.

Es stellt sich die Frage, welche Informationen bleiben, wenn die Euler-Charakteristik verschwindet. Die ersten Ansätze gehen auf Whitehead und Reidemeister zurück, die die Torsion eines azyklischen, d.h. alle Bettizahlen sind null, Kettenkomplexes definierten. Bei deren Ansätze, ist es die Kunst, den Kettenkomplex azyklisch zu bekommen. Für eine spezielle Klasse von 3-Mannigfaltigkeiten geht man dafür über den Funktionskörper $\mathbb{Q}(t)$, wodurch die Torsion eine Funktion wird: das sogenannte getwistete Alexanderpolynom [FV11]. Diese Idee wurde von Li und Zhang im L^2-Setting in [LZ06] aufgegriffen und resultierte in der L^2-Alexander Torsion. In [DFL14] wurden diese Ideen weiter ausgearbeitet.

Für das getwistete Alexanderpolynom haben Friedl und Vidussi in [FV13] einen Zusammenhang zwischen dessen Grad und der Thurston Norm hergestellt. Es ist nun ein Ziel ein analoges Statement für L^2-Alexander Torsion zu erzielen. Die Strategie des Beweises fassen wir hier kurz zusammen. Durch die Arbeiten von Perelmann hat man das folgende Result für irreduzible 3-Mannigfaltigkeiten.

Theorem 0.1. *[AFW15, Geometrization Theorem 1.7.6] Sei N eine kompakte, orientierbare irreduzible 3-Mannigfaltigkeit mit leerem oder torodialen Rand. Dann existiert eine möglischerweise leere Sammlung von disjunkten eingebetteten inkompressiblen Tori $T_1, \dots, T_m$ in N, so dass jede Komponente von N geschnitten entlang $T_1 \cup \dots \cup T_m$ hyperbolisch oder Seifert gefasert ist.*

Sowohl die Thurston Norm, als auch der Grad der L^2-Alexander Torsion verhalten sich additiv unter Verklebung entlang Tori, wodurch sich der Beweis auf Seifert gefaserte Räume und hyperbolische Mannigfaltigkeiten reduzieren lässt. In dieser Arbeit behandeln wir den Fall der Seifert gefaserten Räume (siehe Hauptsatz 7.1).

Aufbau der Arbeit/Kurzzusammenfassung

Die angedachte Leserschaft dieser Arbeit sollte schon mindestens einen Kurs in algebraischer Topologie erfolgreich abgeschlossen haben. So werden Begriffe wie singuläre Homology, CW-Komplexe, universelle Überlagerung und Fundamentalgruppe als bekannt vorrausgesetzt. Darauf aufbauend geben wir im ersten Abschnitt eine kleine Einführung in die Theorie der L^2-Bettizahlen und der L^2-Torsion. Wir beginnen mit den Grundlagen der von Neumann-Gruppenalgebra, die als Koeffizientensystem für die im zweiten Kapitel definierte Homologie für G-CW-Komplexe dient. Kapitel 3 und 4 dienen dem Leser als Einführung in die L^2-Torsion, die zur Vorstellung einer getwisten Version, der L^2-Alexander Torsion, führt. Im darauf folgenden Kapitel 5 findet diese getwistete Version Anwendung im Bereich der Topologie. Im sechsten Kapitel lösen wir uns zunächst von den L^2-Invarianten und machen einen Ausflug in die Theore der Seifert gefaserten Räume. Dort stellen wir einen Zusammenhang zwischen der Thurston Norm und der Orbifaltigkeits-Euler-Charakteristik her. Dieser Zusammenhang ist der Schlüssel zum Beweis des Hauptresultats: die Berechnung der L^2-Alexander Torsion von Seifert gefaserten Räumen. Wer mit L^2-Invarianten vertraut ist kann gleich mit Kapitel 4.3 anfangen.

1 Group von Neumann Algebra

1.1 Group algebra and its completion

In this master thesis every group is countable if not stated otherwise. So let G be a (countable) group. Recall that the group algebra $\mathbb{C}[G]$ is the associative $\mathbb{C}$-algebra with G as a basis and multiplication is defined by using the group multiplication. More precisely we have

$$\mathbb{C}[G] = \left\{ \sum_{x \in G} a_x x \mid a_x \in \mathbb{C} \text{ almost all zero} \right\}.$$

For $\alpha, \beta \in \mathbb{C}[G]$ with $\alpha = \sum a_x x$ and $\beta = \sum b_x x$ addition and multiplication are defined by

$$\alpha + \beta = \left(\sum_{x \in G} a_x x \right) + \left(\sum_{x \in G} b_x x \right) := \sum_{x \in G} (a_x + b_x) x$$

and

$$\alpha\beta = \left(\sum_{x \in G} a_x x \right) \cdot \left(\sum_{y \in G} b_y y \right) := \sum_{x,y \in G} (a_x b_y) xy$$

which can be rewritten to

$$\alpha\beta = \sum_{x,y \in G} (a_{xy^{-1}} b_y) x = \sum_{x,y \in G} (a_x b_{x^{-1}y}) y.$$

Moreover, for $\lambda \in \mathbb{C}$ the scalar multiplication is given by

$$\lambda\alpha = \lambda \left(\sum_{x \in G} a_x x \right) = \sum_{x \in G} (\lambda a_x) x.$$

One easily checks that these operations satisfy all ring axioms and hence $\mathbb{C}[G]$ is a $\mathbb{C}$ algebra. For $g \in G$ we have $1 \cdot g \in \mathbb{C}[G]$, thus we can view G as a subset of $\mathbb{C}[G]$ which is in fact a basis. Next we define an inner product and a corresponding norm on $\mathbb{C}[G]$ by

$$(\alpha, \beta) := \sum_{x \in G} a_x \overline{b}_x,$$

where $\overline{b}_x$ is the complex conjugate of b_x.

$$\|a\| = (\alpha, \alpha)^{1/2} = \sqrt{\sum_{x \in G} |a_x|^2}.$$

Furthermore, we set

$$\overline{\alpha} = \sum_{x \in G} \overline{a_x} x^{-1}.$$

With the previous notation we obtain

Lemma 1.1. *The map $(\ ,\ )$ is a inner product. For all $\alpha, \beta, \gamma \in \mathbb{C}[G]$ we have*

$$(\alpha, \beta\gamma) = (\alpha\overline{\gamma}, \beta) = (\overline{\beta}\alpha, \gamma).$$

Thus, $\overline{\alpha}$ is the adjoint element to α for left and right multiplication.

Proof. The calculation that $(\ ,\ )$ is a hermitian form is left to the reader. Here we just show the second statement. By definition we have

$$\overline{\beta}\alpha = \sum_{x,y\in G} (\overline{b}_{x^{-1}} a_y) xy = \sum_{x,y\in G} (\overline{b}_{xy^{-1}} a_x) y.$$

Therefore

$$(\alpha, \beta\gamma) = \sum_{x,y\in G} a_x \overline{b}_{xy^{-1}} \overline{c}_y = \sum_{x,y\in G} \overline{b}_{xy^{-1}} a_x \overline{c}_y = (\overline{\beta}\alpha, \gamma).$$

In a similar manner:

$$(\alpha, \beta\gamma) = \sum_{x,y\in G} a_x \overline{b}_y \overline{c}_{y^{-1}x} = \sum_{x,y\in G} a_x \overline{c}_{y^{-1}x} \overline{b}_y = (\alpha\overline{\gamma}, \beta).$$

□

Now $\mathbb{C}[G]$ has a pre-Hilbert structure and we write $l^2(G)$ for its completion i.e.

$$l^2(G) = \left\{ \sum_{x\in G} a_x x \mid \sum_{x\in G} |a_x|^2 < \infty \right\}.$$

Note that the multiplication of $\mathbb{C}[G]$ cannot be reasonably continued to $l^2(G)$, so that $l^2(G)$ is not a ring in general. But G is still acting on $l^2(G)$ from left and right and these actions are commuting. Moreover, these actions extend linear, so that $l^2(G)$ is a $\mathbb{C}[G]$-bimodule. We fix this result in the following lemma.

Lemma 1.2. *The completion $l^2(G)$ is a $\mathbb{C}[G]$-bimodule. Let $\ell_\alpha \colon l^2(G) \to l^2(G)$ be the map defined by $x \mapsto \alpha x$ left multiplication for some fixed $\alpha \in \mathbb{C}[G]$, then ℓ_α is bounded by*

$$|\alpha| := \sum_{x\in G} |a_x|.$$

The same results holds for r_α defined analogously for the right multiplication. Further, both G actions are isometric.

Proof. We first show, that the G action is isometric. Recall the definition of the adjoint operator, we have

$$(x\gamma, x\gamma) = (x^* x\gamma, \gamma) = (x^{-1}x\gamma, \gamma) = (\gamma, \gamma).$$

Now for arbitrary $\alpha \in \mathbb{C}[G]$ we conclude

$$\|\alpha\gamma\| = \left\| \sum_{x\in G} a_x x\gamma \right\| \leq \sum_{x\in G} |a_x| \|x\gamma\| = \sum_{x\in G} |a_x| \|\gamma\| = |\alpha| \|\gamma\|.$$

□

1.2 Group von Neumann algebra and Hilbert $\mathcal{N}(G)$-module

Let $\mathfrak{B}(l^2(G))$ denote the set of all bounded endomorphisms of $l^2(G)$. With the previous lemma, we can embed $\mathbb{C}[G]$ in $\mathfrak{B}(l^2(G))$ in two ways by either sending α to r_α or to ℓ_α. A crucial fact about the two embeddings is, that for all $\alpha, \beta \in \mathbb{C}[G]$ we have

$$\ell_\alpha r_\beta = r_\beta \ell_\alpha.$$

This can be taken as the motivation of the following definition.

Definition 1.3. We define the group von Neumann algebra $\mathcal{N}(G)$ as the set

$$\mathcal{N}(G) := \left\{ A \in \mathfrak{B}\left(l^2(G)\right) \mid A\ell_\alpha = \ell_\alpha A \ \forall \alpha \in \mathbb{C}[G] \right\}.$$

In other words the Group von Neumann algebra $\mathcal{N}(G)$ is the set of all G-equivariant bounded operators from $l^2(G)$ to $l^2(G)$, where G-equivariant refers to the left action.

One easily checks that this is a subalgebra of $\mathfrak{B}(l^2(G))$, which is closed under taking adjoint. As already mentioned we have $\mathbb{C}[G] \subset \mathcal{N}(G)$. In addition to that, we can embed $\mathcal{N}(G)$ linear in $l^2(G)$, as we shall see in the next lemma.

Lemma 1.4. *The* $\mathbb{C}$*-linear map*

$$\begin{aligned} \mathrm{ev} \colon \mathcal{N}(G) &\longrightarrow l^2(G) \\ \phi &\longmapsto \phi(1) \end{aligned}$$

is injective and satisfies $\mathrm{ev}(\phi^*) = \overline{\phi(1)}$.

Proof. Let $\phi \in \mathcal{N}(G)$ with $\phi(1) = 0$ and $\gamma \in \mathbb{C}[G]$ with $\gamma = \sum_{i=1}^n c_i x_i$. Then we have

$$\phi(\gamma) = \sum_{i=1}^n c_i \phi(x_i) = \sum_{i=1}^n c_i x_i \phi(1) = 0$$

and since ϕ is continuous and $\mathbb{C}[G]$ is dense in $l^2(G)$ we conclude $\phi \equiv 0$. Next we calculate for an arbitrary $x \in G$

$$\begin{aligned} (\phi(1), x) &= (1, \phi^*(x)) = (1, x\phi^*(1)) \\ &= (x^{-1}, \phi^*(1)) = \overline{(\phi^*(1), x^{-1})}. \end{aligned}$$

Thus,

$$\phi^*(1) = \sum_{x \in G} (\phi^*(1), x) \cdot x = \sum_{x \in G} \overline{(\phi(1), x^{-1})} \cdot x = \overline{\phi(1)}.$$

□

The second part can be seen as a generalization of Lemma 1.1. In addition we have $\mathbb{C}[G]$ is dense in $\mathcal{N}(G)$. Our standard model is $l^2(G)^n$. But we want to consider suitable subspaces.

Definition 1.5. A Hilbert $\mathcal{N}(G)$-module is a Hilbert space V with a linear isometric G-action and which is G-isometric to a G-stable closed subspace of $l^2(G)^n$ for some $n \in \mathbb{N}$. A map between Hilbert $\mathcal{N}(G)$-modules $f: V \to W$ is a bounded G-equivariant operator.

Note that this slightly differs from the definition in [Lüc03], since we only consider finite dimensional Hilbert $\mathcal{N}(G)$-modules.

On the first glance the naming Hilbert $\mathcal{N}(G)$-module seems misleading because in the defintion $\mathcal{N}(G)$ does not appear. But if we have a Hilbert $\mathcal{N}(G)$-morphism $f: l^2(G)^n \to l^2(G)^m$, we can write it as a matrix over $\mathcal{N}(G)$.

Definition 1.6. A Hilbert $\mathcal{N}(G)$-chain complex C_* is a sequence of Hilbert $\mathcal{N}(G)$-modules C_i and Hilbert $\mathcal{N}(G)$-morphisms $c_i : C_i \to C_{i-1}$

$$\dots \xrightarrow{c_{n+1}} C_n \xrightarrow{c_n} C_{n-1} \xrightarrow{c_{n-1}} C_{n-2} \longrightarrow \dots ,$$

such that $c_i \circ c_{i-1} = 0$.

If we want to carry on to homological algebra, we must be careful. Obviously the kernel of a Hilbert $\mathcal{N}(G)$-morphism is a Hilbert $\mathcal{N}(G)$-module. However the image does not need to be closed. Therefore we have to take the closure of the image. We define the homology of a Hilbert $\mathcal{N}(G)$-chain complex C_* by

$$H_n^{(2)}(C_*) = \ker c_n / \overline{\operatorname{Im} c_{n+1}}.$$

Then $H_i^{(2)}(C_*)$ has indeed the structure of a Hilbert $\mathcal{N}(G)$-module as we see in the next lemma.

Lemma 1.7 (Baby L^2-Hodge-de Rham Theorem)**.** *Let C_* be a Hilbert $\mathcal{N}(G)$-chain complex. Define the n-th Laplace operator by*

$$\Delta_n = c_{n+1}c_{n+1}^* + c_n^* c_n.$$

Then we get the orthogonal decomposition

$$C_n = \ker \Delta_n \oplus \overline{\operatorname{Im} c_{n+1}} \oplus \overline{\operatorname{Im} c_n^*}$$

of Hilbert $\mathcal{N}(G)$-modules and the natural map

$$i : \ker c_n \cap \ker c_{n+1} = \ker \Delta_n \to H_n^{(2)}(C_*)$$

is an isometric G-isomorphism.

Proof. We have the decomposition:

$$\begin{aligned} C_p &= \ker c_p^{\perp} \oplus \ker c_p \\ C_p &= \overline{\operatorname{Im} c_{p+1}} \oplus (\operatorname{Im} c_{p+1})^{\perp} = \overline{\operatorname{Im} c_{p+1}} \oplus \ker c_{p+1}^*. \end{aligned}$$

Since $\overline{\operatorname{Im} c_{p+1}} \subset \ker c_p$ we get

$$C_p = \ker c_p^{\perp} \oplus \overline{\operatorname{Im} c_{p+1}} \oplus \left(\ker c_{p+1}^* \cap \ker c_p\right).$$

So we just have to show that $\ker c_{p+1}^* \cap \ker c_p = \ker \Delta_n$. But this follows readily from the calculation:

$$\begin{aligned}(\Delta_n v, v) &= (c_{n+1} c_{n+1}^* v, v) + (c_n^* c_n v, v)\\ &= \|c_{n+1}^* v\|^2 + \|c_n v\|^2.\end{aligned}$$

□

Similar to the homology we define:

Definition 1.8 (weakly exact)**.** A short sequence of Hilbert $\mathcal{N}(G)$-modules

$$0 \longrightarrow U \xrightarrow{\ i\ } V \xrightarrow{\ p\ } W \longrightarrow 0$$

is called weakly exact, if i is injective, p has dense image and $\overline{\operatorname{Im} i} = \ker p$. A morphism $f : V \to W$ is a weak isomorphism, if f is injective and has dense image.

Lemma 1.9. *If $f : V \to W$ is a weak isomorphism, then there exists G-equivarient isometric isomorphism $h : V \to W$.*

Proof. Since f is injective and has dense image, $f^* f : V \to V$ is a strictly positive operator which has dense image. Therefore it exits a self-adjoint operator $g : V \to V$ with $g^2 = f^* f$. We have $\operatorname{Im} g^2 \subset \operatorname{Im} g$ and so g has dense image as well. We define

$$\begin{aligned}\widehat{h} : \operatorname{Im} g &\longrightarrow \operatorname{Im} f\\ v &\longmapsto f g^{-1}(v).\end{aligned}$$

The map $\widehat{h}$ is isometric because of the calculation:

$$\begin{aligned}(\widehat{h}x, \widehat{h}y) &= (f g^{-1} x, f g^{-1} y)\\ &= (f^* f g^{-1} x, g^{-1} y)\\ &= (g^2 g^{-1} x, g^{-1} y)\\ &= (g g^{-1} x, g g^{-1} y) = (x, y),\end{aligned}$$

where we used the fact that g is self-adjoint. Notice that f is invertible if and only if f^* is invertible. This implies $\operatorname{Im} f = W$ iff $\operatorname{Im} g = V$. If f is not surjective we can extend $\widehat{h}$ by continuity to a isometric map $h : V \to W$. The maps f and g are G-equivariant and so is h. The claim follows. □

1.3 The trace map and dimension theorey

Now we are following the work of Eckmann [Eck00]. Since we are mostly dealing with infinite groups and hence $l^2(G)$ has infinite dimension as a vector space, we want to develop a more suitable dimension concept, but with the familiar properties (see Theorem 1.15). For this purpose we introduce the von Neumann trace.

Definition 1.10 (von Neumann trace)**.**

$$\begin{aligned} \operatorname{tr}_{\mathcal{N}(G)} : \mathcal{N}(G) &\longrightarrow \mathbb{C} \\ f &\longmapsto (f(1), 1), \end{aligned}$$

where $1 \in G$ denotes the unit element.

Notice that for an element $\alpha = \sum a_x x \in \mathbb{C}[G]$, we have

$$tr_{\mathcal{N}(G)}(\ell_\alpha) = (\alpha, 1) = a_1$$

which is often referred to as Kaplansky trace. Thus, the von Neumann trace can be seen as an extension of the Kaplansky trace. This map has indeed the basic characteristic of a trace map.

Lemma 1.11. *Let $\varphi, \psi \in \mathcal{N}(G)$. Then we have $\operatorname{tr}_{\mathcal{N}(G)}(\varphi^*) = \overline{\operatorname{tr}_{\mathcal{N}(G)}(\varphi)}$ and the identity*

$$\operatorname{tr}_{\mathcal{N}(G)}(\varphi\psi) = \operatorname{tr}_{\mathcal{N}(G)}(\psi\varphi)$$

holds.

Proof. Let $\varphi(1) = \sum a_x x$ and $\psi(1) = \sum b_x x$. Then

$$\operatorname{tr}_{\mathcal{N}(G)}(\varphi) = (\varphi(1), 1) = a_1 = \overline{(\overline{\varphi(1)}, 1)} = \overline{(\varphi^*(1), 1)} = \overline{\operatorname{tr}_{\mathcal{N}(G)}(\varphi^*)}.$$

A similar calculation yields

$$\begin{aligned} \operatorname{tr}_{\mathcal{N}(G)}(\varphi\psi) &= (\varphi\psi(1), 1) = (\psi(1), \varphi^*(1)) \\ &= (\psi(1), \overline{\varphi(1)}) = \sum_{x \in G} b_x \overline{a_{x^{-1}}} \\ &= \overline{(\overline{\psi(1)}, \varphi(1))} = \overline{(\varphi^*\psi^*(1), 1)} \\ &= \overline{\operatorname{tr}_{\mathcal{N}(G)}((\psi\varphi)^*)} = \operatorname{tr}_{\mathcal{N}(G)}(\psi\varphi). \end{aligned}$$

□

If $F : l^2(G)^n \to l^2(G)^n$ is a morphism, we can write it as a matrix $(F_{ij})_{1 \le i,j \le n}$ with elements of $\mathcal{N}(G)$ as its entries in the usual way by satisfying

$$F(v_1, \ldots, v_n) := \left(\sum_{k=1}^{n} F_{1k} v_k, \ldots, \sum_{k=1}^{n} F_{nk} v_k \right).$$

One observes that $(F_{ij})^*_{1\le i,j\le n} = (F^*_{ji})_{1\le i,j\le n}$. We extend the definition of the trace by

$$\operatorname{tr}_{\mathcal{N}(G)}(F) = \sum_{i=1}^{n} \operatorname{tr}_{\mathcal{N}(G)}(F_{ii})$$

and by Lemma 1.11 we obtain $\operatorname{tr}_{\mathcal{N}(G)}(F) = \operatorname{tr}_{\mathcal{N}(G)}(F^*)$ and $\operatorname{tr}_{\mathcal{N}(G)}(F_1F_2) = \operatorname{tr}_{\mathcal{N}(G)}(F_2F_1)$.

Lemma 1.12. *If $F : l^2(G)^n \to l^2(G)^n$ is self-adjoint and idempotent morphism, then*

$$\operatorname{tr}_{\mathcal{N}(G)}(F) = \sum_{i,j=1}^{n} \|F_{ij}(1)\|^2.$$

Proof.

$$\begin{aligned}
\operatorname{tr}_{\mathcal{N}(G)}(F) &= \sum_{i=1}^{n}(F_{ii}(1), 1) = \sum_{i=1}^{n}(F^2_{ii}(1), 1) \\
&= \sum_{i,j=1}^{n}(F_{ij}F_{ji}(1), 1) = \sum_{i,j=1}^{n}(F_{ij}(1), F^*_{ji}(1)) \\
&= \sum_{i,j=1}^{n}(F_{ij}(1), F_{ij}(1)) = \sum_{i,j=1}^{n} \|F_{ij}(1)\|^2.
\end{aligned}$$

□

Corollary 1.13. *Let $F : l^2(G)^n \to l^2(G)^n$ be idempotent and self-adjoint, then $\operatorname{tr}_{\mathcal{N}(G)} F$ is non negative and $\operatorname{tr}_{\mathcal{N}(G)} F = 0$ implies $F = 0$.*

Now we are ready to define the von Neumann dimension.

Definition 1.14 (von Neumann dimension)**.** Let V be Hilbert $\mathcal{N}(G)$-module. Choose a G-embedding $\varphi : V \to l^2(G)^n$ and the orthogonal G-equivariant projection π onto $\operatorname{Im}\varphi$. Then define the von Neumann dimension via

$$\dim_{\mathcal{N}(G)}(V) = \operatorname{tr}_{\mathcal{N}(G)}(\pi).$$

For $l^2(G)$ the projection is the identity map and we have

$$\dim_{\mathcal{N}(G)} l^2(G) = \operatorname{tr}_{\mathcal{N}(G)} \operatorname{Id} = (1, 1) = 1.$$

However any positive real number can appear as a dimension, as we will see in example 1.20. We gather now the most important properties of the group von Neumann dimension.

Theorem 1.15. *The von Neumann dimension is well defined i.e. independent of the choice of φ and has the following properties:*

1. $\dim_{\mathcal{N}(G)} \ge 0$,

2. $V = 0 \quad \Leftrightarrow \quad \dim_{\mathcal{N}(G)}(V) = 0$,

3. $0 \to U \to V \to W \to 0$ *is a weakly exact sequence of Hilbert* $\mathcal{N}(G)$*-modules, then*

$$\dim_{\mathcal{N}(G)}(U) + \dim_{\mathcal{N}(G)}(W) = \dim_{\mathcal{N}(G)}(V),$$

4. $H < G$ *is a finite index subgroup, then* $\dim_{\mathcal{N}(G)}(V) = \frac{1}{[G:H]} \dim_{\mathcal{N}(H)}(V)$.

Proof. Let $\phi \colon V \to l^2(G)^m$ be another isometric G-embedding and π' the projection on the image of ϕ. Without lost of generality we may assume $m = n$. Otherwise we would extend φ via the map $i \colon l^2(G)^n \to l^2(G)^m$, $z \mapsto (z, 0)$. This will add some rows and columns of zero to the projection π and has no effect on the trace. We get a morphism

$$\begin{aligned} F : \operatorname{Im}\varphi \oplus \operatorname{Im}\varphi^{\perp} &\longrightarrow l^2(G)^n \\ v + w &\longmapsto \phi \circ \varphi^{-1}(v). \end{aligned}$$

Now we use the assumption, that ϕ and φ are isometric, and hence

$$(Fv, Fv) = (v, v) \quad \forall v \in \operatorname{Im}\varphi.$$

Therefore

$$F^*FF^*F(v+w) = F^*Fv = F^*F(v+w)$$

and so F^*F is the orthogonal projection on $\operatorname{Im}\varphi$. And analogously FF^* is the orthogonal projection on $\operatorname{Im}\phi$. We conclude

$$\operatorname{tr}_{\mathcal{N}(G)}(\pi) = \operatorname{tr}_{\mathcal{N}(G)}(F^*F) = \operatorname{tr}_{\mathcal{N}(G)}(FF^*) = \operatorname{tr}_{\mathcal{N}(G)}(\pi').$$

Thus $\dim_{\mathcal{N}(G)}$ is well defined. Now, statement (1) and (2) are direct consequences of Lemma 1.12.
Let $0 \to U \xrightarrow{i} V \xrightarrow{p} W \to 0$ be weakly exact. We have $V = \overline{\operatorname{Im} i} \oplus \ker p^{\perp}$ and $i \colon U \to \overline{\operatorname{Im} i}$, $p^{\perp} \colon \ker p^{\perp} \to W$ are weak isomorphisms. To simplify notion we think of V as a subset of $l^2(G)^n$. Denote the orthogonal projection onto $\overline{\operatorname{Im} i}$ and $\ker p^{\perp}$ with π_1 and π_2. Apparently $\pi_1 \oplus \pi_2$ is the orthogonal projection onto V. We deduce

$$\dim_{\mathcal{N}(G)} V = \operatorname{tr}_{\mathcal{N}(G)} \pi_1 \oplus \pi_2 = \operatorname{tr}_{\mathcal{N}(G)} \pi_1 + \operatorname{tr}_{\mathcal{N}(G)} \pi_2 = \dim_{\mathcal{N}(G)} \overline{\operatorname{Im} i} + \dim_{\mathcal{N}(G)} \ker p^{\perp}.$$

By Lemma 1.9 we have

$$\dim_{\mathcal{N}(G)} U = \dim_{\mathcal{N}(G)} \overline{\operatorname{Im} i} \quad \text{and} \quad \dim_{\mathcal{N}(G)} W = \dim_{\mathcal{N}(G)} \ker p^{\perp}.$$

The last statement is an easy consequence of the fact, that $l^2(G)$ viewed as a Hilbert $\mathcal{N}(H)$-module is isometric isomorph to $l^2(H) \oplus \ldots \oplus l^2(H)$, where the identification is obtained by a choice of representation system of G/H. □

We get a very useful corollary, which characterises weak isomorphisms.

Corollary 1.16. *Let* $f \colon V \to W$ *be a Hilbert* $\mathcal{N}(G)$*-morphism. Then the following statements are equivalent:*

1. *f is a weak isomorphism.*

2. *f is injective and* $\dim_{\mathcal{N}(G)}(V) = \dim_{\mathcal{N}(G)}(W)$.

3. *f has dense image and* $\dim_{\mathcal{N}(G)}(V) = \dim_{\mathcal{N}(G)}(W)$.

4. *the sequence $0 \to V \to W \to 0$ is weakly exact.*

Proof. For (2)⇒(1) we apply Theorem 1.15 (3) to

$$0 \longrightarrow V \longrightarrow W \longrightarrow W/\overline{\operatorname{Im} f} \longrightarrow 0$$

and conclude $\dim_{\mathcal{N}(G)}(W/\overline{\operatorname{Im} f}) = 0$, thus f has dense image by 1.15 (2). For (3)⇒(1) we do the same with the sequence

$$0 \longrightarrow \ker f \longrightarrow V \longrightarrow W \longrightarrow 0$$

and conclude that f is injective. As a consequence of Lemma 1.9: (1)⇒(2),(3). The last implication is true by definition. □

Corollary 1.17. *Let $f : U \to W$ and $g : V \to U$ be weak isomorphism, then $f \circ g$ is weak isomorphism.*

Proof. Obviously the composition of injective maps, is an injective map. We have $\dim_{\mathcal{N}(G)} U = \dim_{\mathcal{N}(G)} W = \dim_{\mathcal{N}(G)} U$. Now $f \circ g$ is a weak isomorphism by 1.16 (2). □

Because it is so important for algebraic topology, we want to mention here the existence of a long weakly exact sequence for a short weakly exact sequence of $\mathcal{N}(G)$-Hilbert chain complexes. The result is due to Cheeger and Gromov [CG85].

Theorem 1.18. *Let $0 \to C_* \to D_* \to E_* \to 0$ be a short weakly exact sequence of Hilbert $\mathcal{N}(G)$-chain complexes, then there exists a long weakly exact sequence*

$$\ldots \to H_n^{(2)}(C_*) \to H_n^{(2)}(E_*) \to H_n^{(2)}(E_*) \to H_{n-1}^{(2)}(C_*) \to \ldots.$$

The theorem, as stated here, becomes wrong, if one allows non finite dimensional Hilbert $\mathcal{N}(G)$-modules. In this case one needs some additional assumptions.

1.4 Examples

Example 1.19. Let G be a finite group. In this case $\mathbb{C}[G]$ is a finite dimensional vector space of dimension $|G|$. All norms are equivalent and we obtain $l^2(G) \cong \mathbb{C}[G] \cong \mathcal{N}(G)$. The von Neumann trace is $1/|G|$ times the ordinary trace for finite vector spaces and the same is true for the von Neumann dimension.

Example 1.20. The Hilbert $\mathcal{N}(\mathbb{Z})$-module $l^2(\mathbb{Z})$ is isometric isomorph to $L^2(S^1)$. The set of all square integrable functions from the S^1 to $\mathbb{C}$. The isomorphism is given by:

$$\begin{aligned} \varphi : l^2(\mathbb{Z}) &\longrightarrow L^2(S^1) \\ \sum_{n\in\mathbb{Z}} a_n t^n &\longmapsto \sum_{n\in\mathbb{Z}} a_n x^n. \end{aligned}$$

We have an $\mathbb{Z}$-action on $L^2(S^1)$ given by

$$\begin{aligned} \mathbb{Z} \times L^2(S^1) &\longrightarrow L^2(S^1) \\ (n, f) &\longmapsto x^n \cdot f(x). \end{aligned}$$

Thus the isomorphism is $\mathbb{Z}$-equivariant. The $\mathcal{N}(\mathbb{Z})$-algebra is given by $L^\infty(S^1)$ as one can see in the following way. Obviously multiplication with a function f from $L^\infty(S^1)$ commutes with the left action. On the other hand, we consider $F \in \mathcal{N}(\mathbb{Z})$. Then $F(1)$, F evaluated on the constant 1 function, is an element in $L^2(S^1)$. One easily sees, that F is given by pointwise multiplication with the function $F(1)$. Since F was bounded we have $\|F\|_{\mathrm{op}} \leq \infty$. But it is well known that in this case $\|F\|_{\mathrm{op}} = \|F(1)\|_\infty$.

Let X be a measurable subset of S^1. The set of all functions $f \in L^2(S^1)$ with support in X are a $\mathcal{N}(\mathbb{Z})$-submodule and the projection is given by multiplication with the characteristic function χ_X. It follows from the definition

$$\dim_{\mathcal{N}(\mathbb{Z})} L^2(X) = \mathrm{tr}_{\mathcal{N}(\mathbb{Z})} \chi_X = \int_{S^1} \chi_X \cdot 1 = \mathrm{Vol}\, X.$$

2 G-CW-Complex

This chapter gives a brief introduction to the concept of a G-CW-complex. In fact we are mostly interested in CW-complex with cellular action (see 2.5). However we will take advantage of the more general definition in 6.29 and 7.2. Our main reference is [Die87], where we have taken most proofs from.

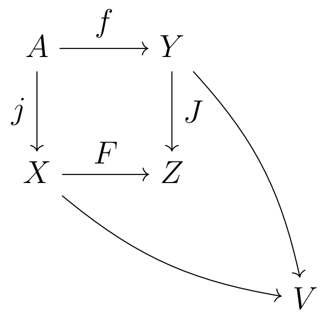
2.1 Definition

Ordinary CW-complexes are constructed by iterated glueing of cells. We transfer this theory to G-spaces. Here G is a locally compact Hausdorff group.

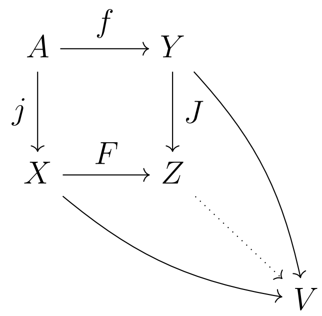
A diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & Y \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle J} \\ X & \xrightarrow{F} & Z \end{array}$$

of G-spaces and G-maps is called G-pushout, if any commutative diagram:



extends uniquely to a diagram:



.

There is a special case of pushout called the adjunction space. Namely if j is an closed embedding. Then we can assume A is a subset of X and consider the following equivalence relation R on the disjoint union $X \sqcup Y$.

$$\begin{aligned}(z_1, z_2) \in R \quad &\Leftrightarrow \quad z_1 = z_2 \\ &\vee \quad (z_1 \in A \wedge z_2 \in A : \quad f(z_1) = f(z_2)) \\ &\vee \quad (z_1 \in A \wedge z_2 \in Y : \quad f(z_1) = z_2) \\ &\vee \quad (z_2 \in A \wedge z_1 \in Y : \quad f(z_2) = z_1)\,.\end{aligned}$$

In this case $Z = X \sqcup Y / \sim_R$, where the topology is induced by the projection $X \sqcup Y \to Z$. This space will be denoted by $Z = X \sqcup_f Y$. The next proposition is the reason, why it is convenient to work with adjunction spaces. It enables us to interpret a pair of spaces by smaller ones.

Proposition 2.1. *If j is a closed embedding. Then J is a closed embedding and the morphism $(F, f)\colon (X, A) \to (X \sqcup_f Y, Y)$ is a relative homomorphism i.e.*

$$X/A \cong X \sqcup_f Y/Y.$$

Proof. We have to check, that J is injective and when $B \subset Y$ is a closed subset, then $J(B)$ is closed. A complete set of representatives is given by $[x] = \{x\}$ for all $x \in X \setminus A$

and $[y] = \{y, j(f^{-1}(y))\}$ for all $y \in Y$. So clearly J is injective. Now we take a closed subset B of Y. By the definition of the topology, we have to show, that $F^{-1}(J(B))$ and $J^{-1}(J(B))$ are closed. The second one is trivial since J is injective. Using the concrete description of the equivalence classes above, one sees $F^{-1}(J(B)) = j(f^{-1}(B))$. Therefore $F^{-1}(J(B))$ is a closed, because f is continuous and j is a closed embedding. □

Definition 2.2 (G-CW-complex)**.** A G-CW-complex X is a G-space together with a G-invariant filtration

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X,$$

such that for any $n \in \mathbb{N}$ there exists a pushout diagram

$$\begin{array}{ccc} \coprod_{i \in J_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_{i \in J_n} q_i} & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{i \in J_n} G/H_i \times D^n & \xrightarrow{\coprod_{i \in J_n} Q_i} & X_n \end{array},$$

where H_i is a closed subgroup of G. X carries the colimit topology with respect to $(X_n)_{n \in \mathbb{N}}$ i.e $B \subset X$ is closed iff $B \cap X_n$ is closed in X_n for all $n \in \mathbb{N}$.

We call a pair $(G/H_i \times D^n, G/H_i \times S^{n-1})$ a cell and refer to (Q_i, q_i) as the characteristic map of the cell. Moreover we say X is of finite type, if $G \backslash X$ is a finite CW-complex and X is a free G-CW-complex, if the action of G is free.

This definition will be applied in two different ways. Firstly, we look at a countable group G with discrete topology and secondly, we use the special case of a S^1-CW-complex. The following discussion belongs to the first application.

2.2 CW-complex with cellular action

Recall the definition of a CW-complex.

Definition 2.3 (CW-complex)**.** A CW-structure of a topological space X is a filtration

$$\emptyset = X_{-1} \subset X_0 \subset \ldots \subset X,$$

such that for any $n \in \mathbb{N}$ there exists a pushout diagram

$$\begin{array}{ccc} \coprod_{i \in J_n} S^{n-1} & \xrightarrow{\coprod_{i \in J_n} q_i} & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{i \in J_n} D^n & \xrightarrow{\coprod_{i \in J_n} Q_i} & X_n \end{array}$$

and X carriers the colimit topology with respect to $(X_n)_{n \in \mathbb{N}}$.

Definition 2.4 (Cellular action)**.** Let X be a G-space and a CW-complex. A group G acts cellular on X if

1. gE is an open cell for every open cell E and $g \in G$,
2. if $gE = E$ then the induced map $g\colon E \to E$ is the identity.

The next theorem characterises G-CW-complexes in terms of cellular action on a CW-complex.

Theorem 2.5. *If G acts cellular on a CW-complex X, then X admits a G-CW-complex structure.*

Proof. Our proof is very close to [Die87, Chapter 2, Proposition 1.15]. We choose a pushout diagram

$$\begin{array}{ccc} \coprod_{i\in J_n} S^{n-1} & \xrightarrow{\coprod_{i\in J_n} q_i} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in J_n} D^n & \xrightarrow{\coprod_{i\in J_n} Q_i} & X_n \end{array}.$$

The group G acts on J_n by permutation, because cellular action implies, that the filtration $\{X_n\}$ is G-invariant. Thus we can partition J_n in orbits $G\alpha_1, G\alpha_2, \ldots$ and choose for each orbit $G\alpha_i$ a G-equivariant bijection $G/H_{\alpha_i} \cong G\alpha_i$. Here H_{α_i} is the isotropy group of α_i i.e. $h\alpha_i = \alpha_i$ for all $h \in H_{\alpha_i}$. We define the map:

$$\begin{aligned} Q_{\alpha_i}\colon G/H_{\alpha_i} \times D^n &\longrightarrow X_n \\ (gH_{\alpha_i}, s) &\longmapsto gQ(\alpha_i, s). \end{aligned}$$

The second condition of cellular action ensures, that it is a well defined map. Namely for any $h \in H_\alpha$, $j \in G_{\alpha_i}$ and $s \in \overset{\circ}{D^n}$:

$$Q(j,s) = hQ(j,s).$$

By continuity this extends to all $s \in D^n$. It remains to proof, that

$$\begin{array}{ccc} \coprod_{\alpha_i\in A} G/H_{\alpha_i} \times S^{n-1} & \xrightarrow{\coprod_{\alpha_i\in A} q_{\alpha_i}} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\alpha_i\in A} G/H_{\alpha_i} \times D^n & \xrightarrow{\coprod_{\alpha_i\in A} Q_{\alpha_i}} & X_n \end{array}$$

is a pushout diagram, which follows directly from the construction and the fact, that a cellular action implies a discrete topology on G. □

Thus we receive the following corollary, which will be the situation in our most cases.

Corollary 2.6. *Let $p\colon Y \to X$ be regular cover of a finite connected CW-complex X. Let G be a subgroup of the deck transformation group. Suppose the action of G is free. Then Y is a free G-CW-complex.*

Proof. A covering induces a CW-structure on Y. The deck transformation group acts cellular on this structure. $\square$

One should notice, that Y doesn't need to be connected. This flexibility will be useful later. Another corollary deals with the restriction to a subgroup. Suppose Y is G-space and $H \subset G$ a subgroup. We can restrict the action of G to elements of H. This yields a H-space $\operatorname{res}_H Y$.

Corollary 2.7. *Let Y be a free G-CW-complex and G a discrete group. Then $\operatorname{res}_H Y$ is H-CW-complex. Moreover, if Y is of finite type and H a finite index subgroup, then $\operatorname{res}_H Y$ is of finite type.*

Proof. Since G is discrete, we can view Y as a CW-complex by forgetting the group structure. Now H acts freely and cellular on Y. $\square$

2.3 L^2-Betti numbers

2.3.1 Homology of G-CW-complexes

Next we want to develop a suitable homology theory for a G-CW-complex X. This is done as follows. The n-chains $C_n(X)$ are defined as $H_n(X_n, X_{n-1})$. The homology is singular homology with $\mathbb{Z}$ coefficients. The boundary map ∂_n is the connecting homomorphism from the long exact sequence of the triple (X_n, X_{n-1}, X_{n-2}). We have:

Theorem 2.8. *Suppose X is a free G-CW-complex of finite type. Then $C_*(X)$ is a free based $\mathbb{Z}[G]$-module, where the basis elements correspond to the characteristic maps.*

Proof. The filtration $\{X_n\}_{n\in\mathbb{N}}$ is G-invariant, so we get a commutative ladder:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_*(X_{n-1}, X_{n-2}) & \longrightarrow & C_*(X_n, X_{n-2}) & \longrightarrow & C_*(X_n, X_{n-1}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & C_*(X_{n-1}, X_{n-2}) & \longrightarrow & C_*(X_n, X_{n-2}) & \longrightarrow & C_*(X_n, X_{n-1}) & \longrightarrow & 0
\end{array}.$$

Now the long exact sequence in homology is natural in the sense, that we obtain the commutative diagram:

$$\begin{array}{ccccccc}
\ldots & \longrightarrow & H_n(X_n, X_{n-1}) & \xrightarrow{\partial} & H_{n-1}(X_{n-1}, X_{n-2}) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} & & \\
\ldots & \longrightarrow & H_n(X_n, X_{n-1}) & \xrightarrow{\partial} & H_{n-1}(X_{n-1}, X_{n-2}) & \longrightarrow & \ldots
\end{array}.$$

Thus the boundary map is indeed a $\mathbb{Z}[G]$-morphism. Next we show that $H_n(X_n, X_{n-1})$ is free. Consider a pushout diagram:

$$\begin{array}{ccc} \coprod_{i\in J_n} G\times S^{n-1} & \xrightarrow{\coprod_{i\in J_n} q_i} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{i\in J_n} G\times D^n & \xrightarrow{\coprod_{i\in J_n} Q_i} & X_n \end{array}.$$

By proposition 2.1 we obtain the following G-homeomorphism

$$X_n/X_{n-1} \;\cong\; \coprod_{i\in J_n} G\times D^n/\coprod_{i\in J_n} G\times S^{n-1} \;\cong\; \bigvee_{i\in J_n} G\times S^n$$

and further

$$H_n(X_n, X_{n-1}) \cong H_n(\bigvee_{i\in J_n} G\times S^n) \cong \bigoplus_{i\in J_n} H_n(G\times S^n).$$

Obviously one has $H_n(G\times S^n)\cong \mathbb{Z}[G]$ as $\mathbb{Z}[G]$-module. □

If we choose a different pushout diagram, we obtain an automorphism of $\bigvee_{i\in J_n} G\times S^n$. So our basis of $\bigoplus_{i\in J_n} H_n(G\times S^n)$ will change by a permutation of the terms of sum or by multiplication with trivial units $\pm g\in G$ in $\mathbb{Z}[G]$.

Definition 2.9 (L^2-Betti numbers)**.** Let X be a free G-CW-complex of finite type. Then we write

$$C_*^{(2)}(X) := l^2(G)\otimes_{\mathbb{Z}[G]} C_*(X)$$

and define the i-th L^2-Betti number

$$b_i^{(2)}(X) := \dim_{\mathcal{N}(G)} H_i^{(2)}(C_*^{(2)}(X)).$$

Next we state and sketch the proofs of the most characteristic properties of the L^2-Betti numbers.

Theorem 2.10 (Homotopy invariance)**.** *Let X, Y be free G-CW-complex of finite type, which are G-homotopic. Then we have*

$$H_i^{(2)}(X) = H_i^{(2)}(Y).$$

for every $i\in\mathbb{Z}$.

Proof. The proof is exactly the same as for ordinary CW-complexes. There are also equivariant analogues of Whitehead theorem, cellular and CW approximation. These can be found in [Mat71]. □

Theorem 2.11 (Characteristic number)**.** *Let H be a finite index subgroup of G. Let X be a G-CW-complex. Then*

$$b_i^{(2)}(X) = [G:H]\cdot b_i^{(2)}(\operatorname{res}_H X).$$

Proof. If we look in the proof of 2.5 how the H-CW-structure is constructed, we conclude for the chain-complex:

$$C_*(\operatorname{res}_H X) \cong \operatorname{res}_H C_*(X).$$

This yields the following description

$$C_*^{(2)}(\operatorname{res}_H X) := l^2(H) \otimes_{\mathbb{Z}[H]} C_*(\operatorname{res}_H X) \cong l^2(H) \otimes_{\mathbb{Z}[H]} \operatorname{res}_H C_*(X).$$

A complete set of representatives $\{g_1, \dots, g_n\}$ defines an H-equivariant isomorphism on generators

$$\begin{aligned} \varphi\colon l^2(H) \otimes_{\mathbb{Z}[H]} \operatorname{res}_H C_*(X) &\longrightarrow \operatorname{res}_H l^2(G) \otimes_{\mathbb{Z}[G]} C_*(X) \\ \alpha \otimes g_i c &\longmapsto \alpha \cdot g_i \otimes c_i, \end{aligned}$$

which has the inverse map

$$\begin{aligned} \operatorname{res}_H l^2(G) \otimes_{\mathbb{Z}[G]} C_*(X) &\longrightarrow l^2(H) \otimes_{\mathbb{Z}[H]} \operatorname{res}_H C_* \\ \alpha g_i \otimes c &\longmapsto \alpha \otimes g_i c. \end{aligned}$$

These are chain isomorphisms and therefore descend to homology. We conclude from 1.15(4):

$$\dim_{\mathcal{N}(G)} H_i^{(2)}(X) = [G:H] \cdot \dim_{\mathcal{N}(H)} H_i^{(2)}(\operatorname{res}_H X).$$

□

2.3.2 L^2-Betti numbers for CW-complexes

Let M be a connected finite CW-complex. Denote $\widetilde{M}$ the universal cover of M. Then $\widetilde{M}$ is also a CW-complex and $\pi_1(M)$ acts freely, transitively and cellular on it. By Theorem 2.5 $\widetilde{M}$ is a $\pi_1(M)$-CW-complex of finite type.

Definition 2.12. Let M be a connected finite CW-complex. Denote $\pi = \pi_1(M)$ and define the L^2-Betti numbers of M by

$$b_i^{(2)}(M) := \dim_{\mathcal{N}(\pi)} H_i^{(2)}(C_*^{(2)}(\widetilde{M})).$$

Example 2.13. When we look at the circle S^1, we have a CW-structure with one 0-cell and one 1-cell. The induced $\mathbb{Z}$-CW-structure is the usual CW-structure of $\widetilde{S^1} \cong \mathbb{R}$ (see Figure 1). The chain complex is given by

$$\begin{aligned} C_0^{(2)}(\widetilde{S^1}) &:= l^2(\mathbb{Z}) \otimes_{\mathbb{Z}[g,g^{-1}]} \mathbb{Z}[g, g^{-1}] \\ C_1^{(2)}(\widetilde{S^1}) &:= l^2(\mathbb{Z}) \otimes_{\mathbb{Z}[g,g^{-1}]} \mathbb{Z}[g, g^{-1}] \end{aligned}$$

with the differential

$$0 \longrightarrow l^2(\mathbb{Z}) \xrightarrow{g-1} l^2(\mathbb{Z}) \longrightarrow 0$$

Since $g-1$ is injective, we have $b_0^{(2)}(S^1) = 0$. By 1.16 $g-1$ has dense image and therefore $b_1^{(2)}(S^1) = 0$. In this example one can see, that the basis of $C_i^{(2)}(\widetilde{M})$ is given by the i-cells of M, which is true in general.

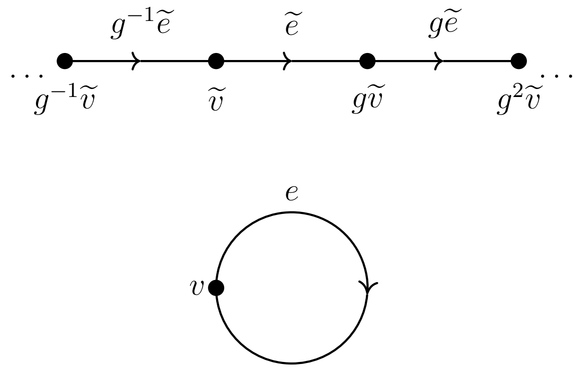


Figure 1: The universal cover $\widetilde{S^1}$ of S^1 with the induced $\langle g \rangle$-CW-structure. The boundary operator is given by $\partial(\widetilde{e}) = g\widetilde{v} - \widetilde{v} = (g-1)\widetilde{v}$

L^2-Betti numbers tend to vanish more often then regular Betti numbers. One reason for this is probably that they are multiplicative under finite covers.

Theorem 2.14. *Let M be a connected finite CW-complex and $\widehat{M}$ a finite cover of degree d. Then we have*

$$d \cdot b_i^{(2)}(M) = b_i^{(2)}(\widehat{M}).$$

Proof. Elementary covering theory tells us that M and $\widehat{M}$ have the same universal cover $\widetilde{M}$ and $\pi_1(\widehat{M})$ is a subgroup of $\pi_1(M)$. So we can apply the multiplicity of Theorem 2.11. □

We get a corollary, which reproofs the calculation of example 2.13.

Corollary 2.15. *Let M be a connected finite CW-complex, which has a selfcovering $p\colon M \to M$ of finite degree $d > 1$. Then all L^2-Betti number, are zero.*

Another very useful property, which is the main obstruction to have vanishing L^2-Betti numbers, is the connection to the Euler characteristic.

Theorem 2.16. *Let M be a connected finite CW-complex. Then the equation*

$$\chi(M) = \sum_{i \in \mathbb{N}} (-1)^i b_i^{(2)}(M)$$

holds.

Proof. Denote $\pi := \pi_1(M)$. The basis for each $C_i^{(2)}(\widetilde{M})$ corresponds to the i-cells of M. Hence:

$$\dim_{\mathcal{N}(\pi)} C_i^{(2)}(\widetilde{M}) = \#i-\text{cells}.$$

Since $\chi(M) = \sum_{i \in \mathbb{N}} (-1)^i \#i-$cells it remains to show

$$\sum_{i \in \mathbb{N}} (-1)^i b_i^{(2)}(M) = \sum_{i \in \mathbb{N}} (-1)^i \dim_{\mathcal{N}(\pi)} C_i^{(2)}(\widetilde{M}),$$

as for the regular Euler characteristic. This is done in exactly the same way. One looks at the two weakly exact sequences:

$$0 \longrightarrow \ker \partial_i \longrightarrow C_i^{(2)}(\widetilde{M}) \longrightarrow \operatorname{Im} \partial_i \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Im} \partial_{i+1} \longrightarrow \ker \partial_i \longrightarrow H_i^{(2)}(\widetilde{M}) \longrightarrow 0$$

and uses the additivity of the von Neumann dimension:

$$\begin{aligned}
\chi(M) &= \sum_{i \in \mathbb{N}} (-1)^i \# i - \text{cells} = \sum_{i \in \mathbb{N}} (-1)^i \dim_{\mathcal{N}(\pi)} C_i^{(2)}(\widetilde{M}) \\
&= \sum_{i \in \mathbb{N}} (-1)^i \left(\dim_{\mathcal{N}(\pi)} \ker \partial_i + \dim_{\mathcal{N}(\pi)} \operatorname{Im} \partial_i \right) \\
&= \sum_{i \in \mathbb{N}} (-1)^i \left(\dim_{\mathcal{N}(\pi)} \operatorname{Im} \partial_{i+1} + \dim_{\mathcal{N}(\pi)} H_i^{(2)}(\widetilde{M}) + \dim_{\mathcal{N}(\pi)} \operatorname{Im} \partial_i \right) \\
&= \sum_{i \in \mathbb{N}} (-1)^i \left(\dim_{\mathcal{N}(\pi)} H_i^{(2)}(\widetilde{M}) \right) = \sum_{i \in \mathbb{N}} (-1)^i b_i^{(2)}(M).
\end{aligned}$$

The last equation holds, because $\dim_{\mathcal{N}(\pi)} \operatorname{Im} \partial_i$ appears twice in the sum, but with different sign. $\square$

2.4 Induction for G-CW-complexes

At the end of this section we construct a new G-CW-complex out of an old one. Suppose X is a H-CW-complex and $i \colon H \to G$ is an injective group homomorphism. Now $G \times X$ has a H-action $(h, (g, x)) = (gh^{-1}, hx)$. We indicate the orbit space with $G \times_H X$. The similarity to the tensor product is non random.

Lemma 2.17. *If X is a free H-CW-complex, then $G \times_H X$ is a free G-CW-complex. Further we have*

$$C_n(G \times_H X) = \mathbb{Z}[G] \otimes_{\mathbb{Z}[H]} C_n(X)$$

Proof. First one has to convince oneself that for any H-map $f \colon X \to Y$ the map $G \times_H f \colon G \times_H X \to G \times_H Y$ sending (g, x) to (g, fx) is continues and hence a G-map. Now it is easy to see, that when $X_{-1} \subset X_0 \subset \ldots \subset X$ is a filtration of X, then $G \times_H X_{-1} \subset G \times_H X_0 \subset \ldots \subset G \times_H X$ is a filtration of $G \times_H X$. Similar for the push out diagram. A H-pushout diagram

$$\begin{array}{ccc}
\coprod_{i \in J_n} H \times S^{n-1} & \xrightarrow{\coprod_{i \in J_n} q_i} & X_{n-1} \\
\big\downarrow & & \big\downarrow \\
\coprod_{i \in J_n} H \times D^n & \xrightarrow{\coprod_{i \in J_n} Q_i} & X_n
\end{array}$$

yields a G-pushout diagram

$$\begin{array}{ccc}
\coprod_{i\in J_n} G\times S^{n-1} & \xrightarrow{\coprod_{i\in J_n} G\times_H q_i} & G\times_H X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i\in J_n} G\times D^n & \xrightarrow{\coprod_{i\in J_n} G\times_H Q_i} & G\times_H X_n
\end{array} .$$

So there is a 1-1 correspondence of characteristic maps of X and $G\times_H X$, which gives the desired isomorphism $C_n(G\times_H X) = \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} C_n(X)$. □

3 Fuglede-Kadison Determinant

3.1 Definition and properties

In this chapter we give the definition of the Fuglede-Kadison determinant. Suppose $f: V\to W$ is a morphism between Hilbert $\mathcal{N}(G)$-modules. We denote $L(f,\lambda)$ the set of all subspaces $L\subset V$, such that $\|f(x)\|\le\lambda\|x\|$ for all $x\in L$. We define the spectral density function

$$\begin{aligned} F(f): \mathbb{R}^+ &\to \mathbb{R}^+ \\ \lambda &\longmapsto \sup\left\{\dim_{\mathcal{N}(G)} L \mid L\in L(f,\lambda)\right\}\end{aligned}$$

Lück showed in [Lüc03, Chapter 3.2], that this function is right continues and hence defines a Borel measure by $F(f)((a,b]) = F(f)(b) - F(f)(a)$. The Fuglede-Kadison determinant is defined by:

$$\det_{\mathcal{N}(G)}(f) := \begin{cases} \exp\left(\int_{0^+}^\infty \ln(\lambda)dF(f)\right), & \int_{0^+}^\infty \ln(\lambda)dF > -\infty, \\ 0, & \text{else.}\end{cases}$$

We say f is of determinant class, when $\int_{0^+}^\infty \ln(\lambda)dF > -\infty$. Next we state the most important properties of $\det_{\mathcal{N}(G)}$, which are proved in [Lüc03, Chapter 3.2].

Proposition 3.1. *Let G be a group and let A, B be matrices over $\mathbb{C}[G]$. Then the following hold.*

1. *Composing A with an isometric isomorphism, doesn't change the Fuglede-Kadison determinant.*

2. *Let A and B be weak isomorphism and of determinant class, then $A\cdot B$ is of determinant class and we have*

$$\det_{\mathcal{N}(G)} A\cdot B = \det_{\mathcal{N}(G)} A\cdot \det_{\mathcal{N}(G)} B.$$

3. *Let $\lambda\in\mathbb{C}\setminus\{0\}$ and set $k=\dim_{\mathcal{N}(G)}\operatorname{Im} A$:*

$$\det_{\mathcal{N}(G)} \lambda A = |\lambda|^k \det_{\mathcal{N}(G)} A.$$

4. *Denote by* $A^\perp\colon \ker A^\perp \to \operatorname{Im} A$ *the induced weak isomorphism of* A. *Then*

$$\det_{\mathcal{N}(G)} A = \det_{\mathcal{N}(G)} A^\perp.$$

5. *If* G *is a subgroup of* H, *then we can view* A *as a matrix over* $\mathbb{C}[H]$ *and*

$$\det_{\mathcal{N}(H)} A = \det_{\mathcal{N}(G)} A$$

.

6.

$$\det_{\mathcal{N}(G)} A \oplus B = \det_{\mathcal{N}(G)} A \det_{\mathcal{N}(G)} B$$

7. *If* A *and* B *are weak isomorphisms and of determinant class. Then* $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ *is of determinant class and*

$$\det_{\mathcal{N}(G)} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det_{\mathcal{N}(G)} A \cdot \det_{\mathcal{N}(G)} B$$

.

8. *If* H *is a subgroup of* G *with finite index* k. *If* A *is an* $m \times n$*-matrix of determinant class. We obtain a* $km \times kn$ *matrix* $\widetilde{A}$ *with entries in* $\mathbb{C}[H]$ *by a choice of complete representation system. For every such choice* $\widetilde{A}$ *is of determinant class and*

$$\det_{\mathcal{N}(G)} A = \det_{\mathcal{N}(H)} \widetilde{A}^{[G:H]}$$

Remark 3.2. If $\varphi\colon G \to H$ is an group isomorphism, then it induces a ring isomorphism $\varphi_*\colon \mathbb{Z}[G] \to \mathbb{Z}[H]$ and an isometric isomorphism between $l^2(G)$ and $l^2(H)$. By the first property $\det_{\mathcal{N}(G)} A = \det_{\mathcal{N}(H)} \varphi_* A$ and by fifth property it is true for φ being a monomorphism.

Most of the properties are familiar from the regular determinant, especially when A and B are weak isomorphisms. But there are some odd phenomenons. For example $\det_{\mathcal{N}(G)}$ is defined even in the case $m \neq n$ and $\det_{\mathcal{N}(G)} 0 = 1$ for the zero map. In general the Fuglede-Kadison determinant is very hard to compute, but in the next section we give a description for the case $G = \mathbb{Z}$.

3.2 Fuglede-Kadison determinant for $\mathcal{N}(\mathbb{Z})$

Lemma 3.3. *Suppose* $f \in \mathcal{N}(\mathbb{Z})$. *Via the isomorphism from example 1.20 we define:*

$$U_\lambda := \left\{ x \in S^1 \mid |f(x)| \leq \lambda \right\} = |f|^{-1}([0, \lambda])$$

and denote

$$V_\lambda := \left\{ g \in L^2(S^1) \mid \operatorname{supp} g \subset U_\lambda \right\}.$$

In this case: $F(f)(\lambda) = \dim_{\mathcal{N}(\mathbb{Z})} V_\lambda = \operatorname{Vol}(U_\lambda)$.

Proof. Let $g \in V_\lambda$. Since $\operatorname{supp} g \subset U_\lambda$ we have

$$|f(x)g(x)| \leq |\lambda g(x)|$$

and hence $V_\lambda \in L(\lambda, f)$. Next we show by contradiction that V_λ is maximal with this property. Suppose there is a Hilbert $\mathcal{N}(\mathbb{Z})$-submodule W, which has a greater dimension. This means, there exists an $u \in W \setminus V_\lambda$ and a measurable set $X \subset S^1 \setminus U_\lambda$ such that $u(x) \neq 0$ for all $x \in X$. Because W is a vector space we can define

$$v := u - \chi_{U_\lambda} u \in W$$

and v is not the null function. We get

$$\|f \cdot v\| > \lambda \cdot \|v\|.$$

This is a contradiction to $v \in W$. Thus, such an u cannot exists and therefore V_λ is the maximal element in $L(f, \lambda)$. □

Theorem 3.4. *Let $f \in L^\infty(S^1)$. Denote with $M_f : L^2(S^1) \to L^2(S^1)$ the multiplication operator with f. Then*

$$\det_{\mathcal{N}(\mathbb{Z})}(M_f) = \exp\left(\int_{S^1} \ln f(|\lambda|) \cdot \chi_{\{z \in S^1 \ | \ f(z) \neq 0\}} dvol_z\right)$$

Proof. Two shorten the notation, we omit the exponential and writing f instead of M_f. The integral $\int_{0^+}^\infty \ln(\lambda) dF(f)$ is defined by an approximation of simple functions of the type $\chi_{\ln^{-1}((a,b])}$. Note that the logarithm is monotone and hence, $\ln^{-1}((a,b]) = (\ln^{-1}(a), \ln^{-1}(b)]$. We calculate

$$\begin{aligned}
\int \chi_{\ln^{-1}((a,b])} dF(f) &:= F(f)(\ln^{-1}(b)) - F(f)(\ln^{-1}(a)) \\
&= \operatorname{Vol} U_{\ln^{-1}(b)} - \operatorname{Vol} U_{\ln^{-1}(a)} \\
&= \operatorname{Vol} |f|^{-1}\left((-\infty, \ln^{-1}(b))\right) - \operatorname{Vol} |f|^{-1}\left((-\infty, \ln^{-1}(a))\right) \\
&= \operatorname{Vol} (\ln \circ |f|)^{-1}([a,b)) \\
&= \int \chi_{(\ln \circ |f|)^{-1}([a,b))} dvol_z.
\end{aligned}$$

Due to the fact that $\ln^{-1}(a) = \exp(a) > 0$ we conclude

$$\begin{aligned}
\det_{\mathcal{N}(\mathbb{Z})}(M_f) &= \exp\left(\int_{0^+}^\infty \ln(\lambda) dF(f)\right) \\
&= \exp\left(\int_{S^1} \ln f(|\lambda|) \cdot \chi_{\{z \in S^1 \ | \ f(z) \neq 0\}} d\mathrm{vol}_z\right).
\end{aligned}$$

□

We are now able to calculate $\det_{\mathcal{N}(\mathbb{Z})} r_\alpha$ for an element $\alpha \in \mathbb{C}[g, g^{-1}]$. We can multiply α with trivial units, because the Fuglede-Kadison determinant is invariant under isometric isomorphisms. Therefore we multiply with the least power of g in α and obtain a polynomial expression.

$$g^k \alpha = \sum_{i=0}^{n} a_i g^i,$$

We set $p_\alpha(x) = \sum_{i=0}^{n} a_i x^i$. The isomorphism from 1.20 sends $g^k\alpha$ to $p_\alpha(x)$. With the preceding discussion we get

$$\begin{aligned}\det_{\mathcal{N}(G)} r_\alpha = \det_{\mathcal{N}(G)} r_{g^k} \cdot r_\alpha &= \exp\left(\int_{S^1} \ln|p_\alpha(\lambda)| \cdot \chi_{\{z\in S^1 \ | \ f(z)\neq 0\}} d\mathrm{vol}_z\right) \\ &= \exp\left(\int_0^1 \ln|p_\alpha(e^{2\pi i t})| dt\right).\end{aligned}$$

The last expression is well known and has been widely studied under the name Mahler measure. Let $z_1, \ldots, z_n$ be the roots of p_α. It can be shown using Jensen's formula that

$$\exp\left(\int_0^1 \ln|p_\alpha(e^{2\pi i t})| dt\right) = |a_0| \prod_{|z_i|\leq 1} \frac{1}{|z_i|} = |a_n| \cdot \prod_{i=1}^{n} \max\{1, |z_i|\}.$$

3.3 Being of determinant class

This subsection is devoted to the question, which morphisms are of determinant class. The starting point of the question is the following conjecture.

Conjecture 3.5. *Let $A \in M(m \times n, \mathbb{Q}[G])$. Then the induced map $r_A : l^2(G)^m \to l^2(G)^n$ is of determinat class.*

For a very large class of group, called sofic group, it has been verified by Elek and Szabó [ES05]. These class is closed under taking subgroups and finite index extension and contains all residually finite groups. By a result of Hempel [Hem87] the fundamental group of a 3-manifold is residually finite. Therefore we have

Proposition 3.6. *Let M be a three manifold. Denote $\pi := \pi_1(M)$. Given a matrix $A \in M(m \times n, \mathbb{Q}[\pi])$. Then the induced map r_A is of determinant class.*

This is the case we are mostly interested in. The next two lemmas are preparation for Section 4 and state that being of determinant class is well behaved under commutative diagrams of weak isomorphism.

Lemma 3.7. *Given a commutative square.*

$$\begin{array}{ccc} U & \xrightarrow{g} & V \\ \downarrow k & & \downarrow h \\ W & \xrightarrow{f} & Z \end{array}$$

if three are weak isomorphism and of determinant class, then the fourth is a weak isomorphism and of determinant class.

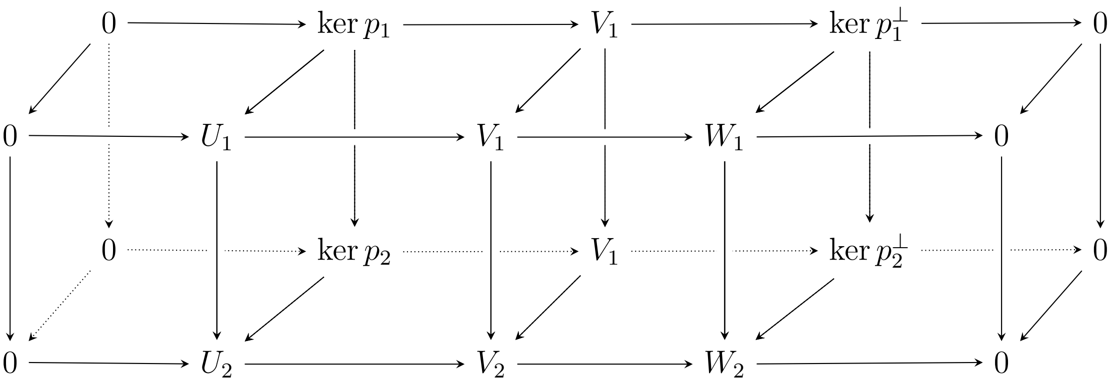
Proof. Without lost of generality the maps g, h are weak isomorphism and of determinant class. Now $h \circ g$ is a weak isomorphism and of determinant class by Corollary 1.17 and Proposition 3.1(2). Since the diagram is commuting the same holds for $k \circ f$. By assumption one of k and f is a weak isomorphism and we conclude by the same argument, that the other map is a weak isomorphism and of determinant class. $\square$

Lemma 3.8. *Given a commutative ladder of Hilbert $\mathcal{N}(G)$-modules:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & V_1 & \longrightarrow & W_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & U_2 & \longrightarrow & V_2 & \longrightarrow & W_2 & \longrightarrow & 0 \end{array}$$

with weakly exact rows and every map on the row is of determinant class. When two of the maps f, g, h are weak isomorphism and of determinant class, then also the third will be a weak isomorphism and of determinant class.

Proof. We look the diagram:



By Lemma 3.7 we can restrict to the case of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \ker p_1 & \longrightarrow & V_1 & \longrightarrow & \ker p_1^\perp & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g|_{\ker p_1}} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g|_{\ker p_1^\perp}} & & \\ 0 & \longrightarrow & \ker p_2 & \longrightarrow & V_2 & \longrightarrow & \ker p_2^\perp & \longrightarrow & 0 \end{array}$$

By property 3.1(6) applied to $g|_{\ker p_1} \oplus g|_{\ker p_1^\perp} = g$ one sees that all three maps are of determinant class. We write $g = (g|_{\ker p_1} \oplus \mathrm{Id}) \circ (\mathrm{Id} \oplus g|_{\ker p_1^\perp})$ and conclude that all three maps are weak isomorphisms. $\square$

4 Algebraic L^2-torsion

In this section we develop the main algebraic tools for the L^2-Alexander torsion, which we want to study in Section 5. We are still following [Lüc03]. But we restrict ourself to the case, where C_* is weakly acyclic.

4.1 Definition of L^2-torsion

Definition 4.1 (L^2-torsion)**.** We say a Hilbert $\mathcal{N}(G)$-complex C_* :

$$0 \longrightarrow \ldots \longrightarrow C_n \xrightarrow{c_n} C_{n-1} \xrightarrow{c_{n-1}} \ldots \longrightarrow C_0 \longrightarrow 0$$

is L^2*-Det-acyclic*, if every morphism c_n is of determinant class and all L^2-Betti number are vanishing. Then the L^2-Torsion is given by

$$\tau^{(2)}(C_*) := \prod_{p\in\mathbb{Z}} \det_{\mathcal{N}(G)}(c_p)^{(-1)^p} \in (0,\infty).$$

If C_* is not L^2-Det-acyclic, we set $\tau^{(2)}(C_*) = 0$. For example, if the Euler characteristic is not zero.

4.2 Two basic properties

Most statements about torsion can be broken down to two formulas: the transformation formula and the multiplicity of torsion. Both formulas will be proved by induction and the next lemma is the inductive step.

Lemma 4.2. *Let C_* be a L^2-Det-acyclic chain complex of length $n+1$. We define a sub-chain complex $C_*^{'}$ by*

$$C_*^{'} : \ldots \longrightarrow 0 \longrightarrow C_{n+1} \longrightarrow \overline{\operatorname{Im} c_{n+1}} \longrightarrow 0 \longrightarrow \ldots$$

and $C_*^{''} := C_*/C_*^{'}$. Then the identity

$$\tau^{(2)}(C_*^{'})\tau^{(2)}(C_*^{''}) = \tau^{(2)}(C_*)$$

holds.

Proof. By construction C_*' and C_*'' are L^2-Det-acyclic. We obtain a commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & C_{n+1} & \longrightarrow & C_{n+1} & \longrightarrow & 0 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle c_{n+1}} & & \downarrow{\scriptstyle c_{n+1}} & & \downarrow & & \downarrow \\
0 & \longrightarrow & \overline{\operatorname{Im} c_{n+1}} & \longrightarrow & C_n & \longrightarrow & (\operatorname{Im} c_{n+1})^{\perp} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle c_n^{\perp}} & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & C_{n-1} & \longrightarrow & C_{n-1} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle c_{n-1}} & & \downarrow{\scriptstyle c_{n-1}} & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}
$$

It is $\det_{\mathcal{N}(G)}(c_n) = \det_{\mathcal{N}(G)}(c_n^{\perp})$ by Proposition 3.1(4) and therefore

$$
\begin{aligned}
\tau^{(2)}(C_*^{'})\tau^{(2)}(C_*^{''}) &= \det_{\mathcal{N}(G)}(c_{n+1})^{(-1)^{n+1}} \cdot \det_{\mathcal{N}(G)}(c_n^{\perp})^{(-1)^n} \prod_{i=0}^{n-1} \det_{\mathcal{N}(G)}(c_i)^{(-1)^i} \\
&= \tau^{(2)}(C_*)
\end{aligned}
$$

□

Theorem 4.3 (Transformation formula)**.** *Let C_* or D_* are L^2-Det-acyclic and $f_* \colon C_* \to D_*$ be a weak chain-isomorphism. Then both are L^2-Det-acyclic and we have*

$$
\tau^{(2)}(D_*) = \left(\prod_{p \in \mathbb{Z}} \det_{\mathcal{N}(G)} (f_p)^{(-1)^{p+1}} \right) \tau^{(2)}(C_*)
$$

Proof. The proof is done by induction over the length of the complex C_*. Making the base case $n = 2$ will help us with the inductive step. We obtain the following commutative diagram.

$$
\begin{array}{ccccccc}
0 & \longrightarrow & C_1 & \xrightarrow{c_1} & C_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow \\
0 & \longrightarrow & D_1 & \xrightarrow{d_1} & D_0 & \longrightarrow & 0
\end{array}
$$

Now the maps c_1 and d_1 are weak isomorphisms, because the chain complexes are weakly acyclic. Therefore we can calculate:

$$
\begin{aligned}
\det_{\mathcal{N}(G)} (f_0 c_1) &= \det_{\mathcal{N}(G)}(d_1 f_1) \\
\det_{\mathcal{N}(G)}(c_1) \cdot \det_{\mathcal{N}(G)}(f_0) &= \det_{\mathcal{N}(G)}(f_1) \cdot \det_{\mathcal{N}(G)}(d_1) \\
\tau^{(2)}(D_*) &= \frac{\det_{\mathcal{N}(G)} f_1}{\det_{\mathcal{N}(G)} f_0} \cdot \tau^{(2)}(C_*).
\end{aligned}
$$

Let the statement be true for $n > 2$. Suppose C_* has length $n+1$. We define the chain complexes C'_* and C''_* respectively D'_*, D''_* as in Lemma 4.2. Now the statement is true for C'_* and D'_* by the base case. Moreover it is true for C''_* and D''_* by induction hypothesis. We obtain the equations

$$\tau^{(2)}(D'_*) = \det_{\mathcal{N}(G)}(f_{n+1})^{(-1)^{n+2}} \cdot \det_{\mathcal{N}(G)}(f'_n)^{(-1)^{n+1}} \tau^{(2)}(C'_*)$$
$$\tau^{(2)}(D''_*) = \det_{\mathcal{N}(G)}(f''_n)^{(-1)^{n+1}} \cdot \prod_{i=0}^{n-1} \det_{\mathcal{N}(G)}(f_i)^{(-1)^{i+1}} \tau^{(2)}(C''_*)$$

and the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \overline{\operatorname{Im} c_{n+1}} & \longrightarrow & C_n & \longrightarrow & (\operatorname{Im} c_{n+1})^{\perp} & \longrightarrow & 0 \\ & & \downarrow f'_n & & \downarrow f_n & & \downarrow f''_n & & \\ 0 & \longrightarrow & \overline{\operatorname{Im} d_{n+1}} & \longrightarrow & D_n & \longrightarrow & (\operatorname{Im} d_{n+1})^{\perp} & \longrightarrow & 0 \end{array}$$

One easily sees, that $f'_n \oplus f''_n = f_n$ and hence

$$\det_{\mathcal{N}(G)}(f'_n) \cdot \det_{\mathcal{N}(G)}(f''_n) = \det_{\mathcal{N}(G)}(f_n).$$

The equation of Lemma 4.2 finishes the proof. □

Theorem 4.4 (Product formula)**.** *Let* $0 \longrightarrow C_* \xrightarrow{i_*} D_* \xrightarrow{p_*} E_* \longrightarrow 0$ *be a weakly short exact sequence of* $\mathcal{N}(G)$*-chain complexes. Suppose for every* $n \in \mathbb{N}$*, the morphism* $i_n\colon C_n \to \ker p_n$ *and* $p_n^{\perp}\colon (\ker p_n)^{\perp} \to E_n$ *are isometric. If two out of the three* C_*, D_*, E_* *are* L^2*-Det-acyclic, then the third is* L^2*-Det-acyclic and we get*

$$\tau^{(2)}(C_*)\tau^{(2)}(E_*) = \tau^{(2)}(D_*)$$

Proof. The maps $i_*\colon C_* \to \ker p_*$ and $p_*^{\perp}\colon (\ker p_*)^{\perp} \to E_*$ are weak isometric chain isomorphisms and therefore $\det_{\mathcal{N}(G)} i_n = \det_{\mathcal{N}(G)} p_n = 1$. The transformation formula 4.3 yields:

$$\tau^{(2)}(\ker p_*) = \tau^{(2)}(C_*)$$
$$\tau^{(2)}(E_*) = \tau^{(2)}((\ker p_*)^{\perp}).$$

Thus it is sufficient to show the theorem for $C_* = \ker p_*$ and $E_* = (\ker p_*)^{\perp}$. In this case we can write the differential d_n of D_* by

$$d_n = \begin{pmatrix} d'_n & \overline{d_n} \\ 0 & d''_n \end{pmatrix} : \ \ker q_n \oplus \ker q_n^{\perp} \to \ker q_{n-1} \oplus (\ker q_{n-1})^{\perp},$$

where d'_n and d''_n are the differentials of $\ker q_n$ and $\ker q_n^{\perp}$. Again we use induction on the length of D_* starting with $n = 2$. For $n = 2$ we can apply Lemma 3.8 and deduce

$$\det_{\mathcal{N}(G)}(d_n) = \det_{\mathcal{N}(G)} \begin{pmatrix} d'_1 & \overline{d_1} \\ 0 & d''_1 \end{pmatrix} = \det_{\mathcal{N}(G)}(d'_1) \det_{\mathcal{N}(G)}(d''_1).$$

With the notation from Lemma 4.2, we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C'_* & \longrightarrow & C_* & \longrightarrow & C''_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D'_* & \longrightarrow & D_* & \longrightarrow & D''_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E'_* & \longrightarrow & E_* & \longrightarrow & E''_* & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and the equations

$$\tau^{(2)}(C'_*)\tau^{(2)}(C''_*) = \tau^{(2)}(C_*) \tag{1}$$
$$\tau^{(2)}(D'_*)\tau^{(2)}(D''_*) = \tau^{(2)}(D_*) \tag{2}$$
$$\tau^{(2)}(E'_*)\tau^{(2)}(E''_*) = \tau^{(2)}(E_*). \tag{3}$$

By induction hypotheses we obtain

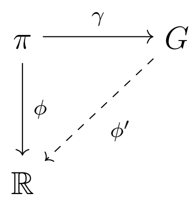
$$\tau^{(2)}(C'_*)\tau^{(2)}(E'_*) = \tau^{(2)}(D'_*) \tag{4}$$
$$\tau^{(2)}(C''_*)\tau^{(2)}(E''_*) = \tau^{(2)}(D''_*) \tag{5}$$

All equations combined proof the claim. □

4.3 Hilbert $\mathcal{N}(G)$-complex for a compatible duo

In this section we look at twisted L^2-invariants. The ideas come from the paper of Friedl, Lück and Dubois [DFL14] about L^2-Alexander torsion. Here we work out the algebraic fundamentals.

Definition 4.5. Let π be a group, $\phi \in \mathrm{Hom}(\pi, \mathbb{R})$, and $\gamma\colon \pi \to G$ a homomorphism. We say that ϕ and γ are compatible, if $\phi\colon \pi \to \mathbb{R}$ factors through γ i.e. there exists $\phi'\colon G \to \mathbb{R}$ such that $\phi = \phi'\gamma$.



For $t \in \mathbb{R}^+$ we define the ring homomorphism

$$\begin{aligned}\kappa(\phi,\gamma,t)\colon \mathbb{Z}[\pi] &\longrightarrow \mathbb{R}[G]\\ \sum_{x\in G} a_x x &\longmapsto \sum_{x\in G} a_x t^{\phi(x)}\gamma(x).\end{aligned}$$

It lets us view $\mathbb{R}[G]$ and hence $\mathcal{N}(G)$ as a $\mathbb{Z}[\pi]$-right module. This gives rise to a functor from finite free $\mathbb{Z}[\pi]$-modules to Hilbert $\mathcal{N}(G)$-modules.

$$M_*^{\phi,\gamma,t} := l^2(G) \otimes_{\mathbb{R}[G]} \mathbb{R}[G] \otimes_{\kappa(\phi,\gamma,t)} M.$$

When we choose a basis such that $M \cong \mathbb{Z}[\pi]^n$, we get an isomorphism

$$M^{\phi,\gamma,t} = l^2(G)^n$$

and the structure of Hilbert $\mathcal{N}(G)$-module. We write

$$\tau^{(2)}\left(C_*,\phi,\gamma,t\right) := \tau^{(2)}(C_*^{\phi,\gamma,t})$$

for the corresponding torsion. However all this depends on the choice of basis. Therefore we look at based[1] $\mathbb{Z}[\pi]$-chain complexes C_*. A useful fact is, that this functor is commuting with direct sums. Thus this functor is exact. Let $F : \mathbb{Z}[\pi]^n \to \mathbb{Z}[\pi]^m$ be a $\mathbb{Z}[\pi]$-morphism of based $\mathbb{Z}[\pi]$-modules. We have a basis for the source and target, so we can write F as a matrix $(F_{ij})_{\substack{1\le i\le n\\ 1\le j\le m}}$, which multiplies by right. We can look at:

$$\rho(F,\phi,\gamma,t) := \left(\kappa(\phi,\gamma,t)(F_{ij})\right)_{\substack{1\le i\le n\\ 1\le j\le m}}.$$

One easily checks, that the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{Z}[\pi]^m)^{\phi,\gamma,t} & \xrightarrow{F^{\phi,\gamma,t}} & (\mathbb{Z}[\pi]^n)^{\phi,\gamma,t} \\ \downarrow & & \downarrow \\ l^2(G)^n & \xrightarrow{\rho(F,\phi,\gamma,t)} & l^2(G)^m \end{array}.$$

This leads to a crucial observation about torsion. Namely given a based $\mathbb{Z}[\pi]$-chain complex C_*, then all differential c_i can be seen as matrices and we get

$$\tau^{(2)}\left(C_*,\phi,\gamma,t\right) = \tau^{(2)}(C_*^{\phi,\gamma,t}) = \prod_{i\in Z} \det\nolimits_{\mathcal{N}(G)}\left(\rho(c_i,\phi,\gamma,t)\right)^{(-1)^i}.$$

If we want to understand $\tau^{(2)}\left(C_*,\phi,\gamma,t\right)$, it will be sufficient to look at the differentials c_i, the morphism $\kappa(\phi,\gamma,t)$, and how its effect the Fuglede-Kadison determinant. The next lemma is an example of this idea.

[1]Here based means finite and free.

Lemma 4.6. *Let C_* be a free $\mathbb{Z}[\pi]$-chain complex and $\varphi\colon G \to K$ be a monomorphism. Suppose ϕ and $\varphi \circ \gamma$ are compatible. Then $\phi\colon \pi \to \mathbb{R}$ and $\gamma\colon \pi \to G$ are compatible and*

$$\tau^{(2)}(C_*, \phi, \varphi\gamma, t) = \tau^{(2)}(C_*, \phi, \gamma, t).$$

Proof. First we observe for a trivial unit $h \in \mathbb{Z}[\pi]$, that

$$\kappa(\phi, \varphi \circ \gamma, t)(h) = t^{\phi(h)} \varphi\gamma(h) = \varphi_* \left(t^{\phi(h)}\gamma(h)\right) = \varphi_* \left(\kappa(\phi, \gamma, t)(h)\right).$$

Everything extends linear and we conclude:

$$\begin{aligned} \tau^{(2)}(C_*, \phi, \varphi\gamma) &= \prod_{i \in Z} \det_{\mathcal{N}(K)} \left(\rho(c_i, \phi, \varphi\gamma, t)\right)^{(-1)^i} \\ &= \prod_{i \in Z} \det_{\mathcal{N}(K)} \left(\varphi_* \left(\rho(c_i, \phi, \gamma, t)\right)\right)^{(-1)^i} \\ &= \prod_{i \in Z} \det_{\mathcal{N}(G)} \left(\left(\rho(c_i, \phi, \gamma, t)\right)\right)^{(-1)^i}. \end{aligned}$$

The last equation is obtained from the remark after Proposition 3.1. □

In view of the transformation formula we emphasize again, that the torsion depends on the choice of basis. One could be more careful and consider ordered basis, but:

Proposition 4.7. *Let C_* be a free $\mathbb{Z}[\pi]$-chain complex with ordered basis and $\phi\colon \pi \to \mathbb{R}$, $\gamma\colon G \to \mathbb{R}$ are compatible. Denote by $\widetilde{C_*}$ the free $\mathbb{Z}[\pi]$-chain complex which rises from C_* by a permutation of basis. Then we have*

$$\tau^{(2)}(C_*, \phi, \gamma, t) = \tau^{(2)}(\widetilde{C_*}, \phi, \gamma, t).$$

Proof. The isomorphism $f_*\colon C_* \to \widetilde{C_*}$ is given by a permutation matrix A_{f_i}. Obviously $\rho(A_{f_i}, \phi, \gamma, t)$ is also a permutation matrix and we have

$$\det_{\mathcal{N}(G)}(\rho(A_{f_i}, \phi, \gamma, t)) = 1,$$

because permutation matrices are isometrics. The transformation formula states

$$\begin{aligned} \tau^{(2)}(C_*, \phi, \gamma, t) &= \prod_{p \in \mathbb{Z}} \det_{\mathcal{N}(G)}(\rho(A_{f_p}, \phi, \gamma, t))^{(-1)^{p+1}} \tau^{(2)}(\widetilde{C_*}, \phi, \gamma, t) \\ &= \tau^{(2)}(\widetilde{C_*}, \phi, \gamma, t). \end{aligned}$$

□

4.4 Restriction and Induction

Suppose C_* is a based $\mathbb{Z}[\pi]$-chain complex. Let H be a finite index subgroup of π. We can view C_* as $\mathbb{Z}[H]$-chain complex by restricting the action to H. We denote the resulting complex by $\operatorname{res}_H C_*$. Obviously this is a free chain complex. To compute $\tau^{(2)}(\operatorname{res}_H C_*, \phi, \gamma, t)$ we would like to use Proposition 3.1(8). However this is not possible without restrictions to the map γ. For example if γ is the trivial map, we loose all information about H. This problem doesn't appear if $\ker\gamma \subset H$. Then isomorphism theorems yield:

$$\pi/H \cong (\pi/\ker\gamma)\,/\,(H/\ker\gamma) \cong \operatorname{Im}\gamma/\operatorname{Im}\gamma|_H.$$

We obtain the following proposition.

Proposition 4.8 (restriction)**.** *Let C_* be a based $\mathbb{Z}[\pi]$-chain complex. Let H be a finite index subgroup. Suppose $\phi\colon \pi \to \mathbb{R}$ and $\gamma\colon \pi \to \Gamma$ are compatible with* $\ker\gamma \subset H$*. Then there exists $r \in \mathbb{R}$ such that:*

$$\tau^{(2)}(C_*, \phi, \gamma, t) = t^r \cdot \tau^{(2)}(\operatorname{res}_H C_*, \phi, \gamma, t)^{[\pi:H]}$$

This result is best possible in the sense, that we have to choose a basis for $\operatorname{res}_H C_*$ and for any two reasonable choices the torsion differs by a factor t^r for some $r \in \mathbb{R}$. By reasonable choice we mean a choice of isomorphism of Theorem 1.15(4). We will see this phenomenon again when we deal with the topological application.

Proof. Denote $\Gamma_H = \operatorname{Im}\gamma|_H$. The proofs consists of two parts. First we show

$$\operatorname{res}_{\Gamma_H} l^2(\Gamma) \otimes_{\kappa(\phi,\gamma,t)} C_* \cong l^2(\Gamma_H) \otimes_{\phi,\gamma} \operatorname{res}_H C_*.$$

Afterwards we can use the transformation formula to calculate the torsion. Then the claim follows from Proposition 3.1. We have already seen that we can restrict to the case, where γ is surjective. Let $\{g_1, \ldots, g_n\}$ be a complete set of representatives of π/H. This yields a complete set of representatives y_i of Γ_H in Γ via $\gamma(g_i) = y_i$. We define the chain isomorphism on generators

$$\begin{aligned} f_*\colon \operatorname{res}_{\Gamma_H} l^2(\Gamma) \otimes_{\kappa(\phi,\gamma,t)} C_* &\longrightarrow l^2(\Gamma_H) \otimes_{\kappa(\phi,\gamma,t)} \operatorname{res}_H C_* \\ y_i \otimes c &\longmapsto 1 \otimes g_i c \end{aligned}$$

with the inverse map:

$$\begin{aligned} f_*^{-1}\colon l^2(\Gamma_H) \otimes_{\kappa(\phi,\gamma,t)} \operatorname{res}_H C_* &\longrightarrow \operatorname{res}_{\Gamma_H} l^2(\Gamma) \otimes_{\kappa(\phi,\gamma,t)} C_* \\ 1 \otimes g_i c &\longmapsto y_i \otimes c. \end{aligned}$$

Next we have to show, that the determinant of f_i is a power of t for every $i \in \mathbb{N}$. Let $\{c_1, \ldots, c_r\}$ be the basis of C_i. We get the corresponding basis $\{1 \otimes c_1, \ldots, 1 \otimes c_r\}$ for $l^2(\Gamma) \otimes C_i$ and the basis $\{y_1 \otimes c_1, \ldots, y_n \otimes c_r\}$ for $\operatorname{res}_{\Gamma_H} l^2(\Gamma) \otimes C_i$. On the otherhand we get the basis $\{1 \otimes g_1 c_1, \ldots, 1 \otimes g_n c_r\}$, which is the same as $\{y_1 t^{\phi(g_1)} \otimes c_1, \ldots, y_r t^{\phi(g_n)} \otimes c_r\}$. Hence f_i is a diagonal matrix with $t^{\phi(g_j)}$ as its entries. So it is for some $r_i \in \mathbb{R}$:

$$\det_{\mathcal{N}(G)}(f_i) = t^{r_i}.$$

We deduce by the transformation formula and Proposition 3.1(8):

$$\begin{aligned}\tau^{(2)}(l^2(\Gamma_H)\otimes_{\kappa(\phi,\gamma,t)}\operatorname{res}_H C_*) &= \prod_{i\in\mathbb{Z}} \det_{\mathcal{N}(G)}(f_i)^{(-1)^{i+1}}\cdot\tau^{(2)}(\operatorname{res}_{\Gamma_H} l^2(\Gamma)\otimes_{\kappa(\phi,\gamma,t)} C_*)\\ &= t^r\cdot\tau^{(2)}(\operatorname{res}_{\Gamma_H} l^2(\Gamma)\otimes_{\kappa(\phi,\gamma,t)} C_*)\\ &= t^r\cdot\tau^{(2)}(l^2(\Gamma)\otimes_{\kappa(\phi,\gamma,t)} C_*)^{[\Gamma:\Gamma_H]}.\end{aligned}$$

□

Next we deal with a converse situation. Therefore we need a basic fact about tensor products.

Lemma 4.9. *Let M be a $Z[H]$-module. Moreover let $\phi\colon H\to K$ and $\varphi\colon K\to G$ be a group homomorphisms. Then there is an isomorphism*

$$\mathbb{Z}[G]\otimes_\varphi \mathbb{Z}[K]\otimes_\phi M\cong \mathbb{Z}[G]\otimes_{\varphi\circ\phi} M$$

as left $\mathbb{Z}[G]$-modules.

Proof. The isomorphism is given by the map

$$\begin{aligned} f\colon \mathbb{Z}[G]\otimes_{\varphi\circ\phi} M &\longrightarrow \mathbb{Z}[G]\otimes_\varphi \mathbb{Z}[K]\otimes_\phi M\\ \alpha\otimes m &\longmapsto \alpha\otimes 1\otimes m,\end{aligned}$$

which has the inverse map

$$\begin{aligned} f^{-1}\colon \mathbb{Z}[G]\otimes_\varphi \mathbb{Z}[K]\otimes_\phi M &\longrightarrow \mathbb{Z}[G]\otimes_{\varphi\circ\phi} M\\ \alpha\otimes\beta\otimes m &\longmapsto \alpha\varphi(\beta)\otimes m.\end{aligned}$$

□

In the case $M=\mathbb{Z}[H]$ we get the induced isomorphisms

$$\mathbb{Z}[G]\otimes\mathbb{Z}[K]\otimes\mathbb{Z}[H]\cong\mathbb{Z}[G]\quad\text{and}\quad\mathbb{Z}[G]\otimes\mathbb{Z}[H]\cong\mathbb{Z}[G]$$

both sending an element $\sum a_h\cdot h\in\mathbb{Z}[H]$ to $\sum a_h\cdot\varphi\phi(h)\in\mathbb{Z}[G]$. So we will think about it as equality. We rewrite this observation:

Proposition 4.10. *Let $f\colon H\to\pi$ be a group homomorphism and ϕ, γ compatible. If C_* is a based $\mathbb{Z}[H]$-chain complex, then $D_*=\mathbb{Z}[\pi]\bigotimes_f C_*$ will be a based $\mathbb{Z}[\pi]$-chain complex and we have*

$$C_*^{\phi f,\gamma f,t}=D_*^{\phi,\gamma,t}$$

as $\mathcal{N}(G)$-chain complexes. Therefore we conclude

$$\tau^{(2)}\left(C_*,\phi f,\gamma f,t\right)=\tau^{(2)}\left(D_*,\phi,\gamma,t\right).$$

Proof. By definition we have

$$D_*^{\phi,\gamma,t}\cong l^2(G)\otimes_{\mathbb{R}[G]}\mathbb{R}[G]\otimes_{(\phi,\gamma,t)}\mathbb{Z}[\pi]\otimes_f C_*$$

and

$$C_*^{\phi f,\gamma f,t}=l^2(G)\otimes_{\mathbb{R}[G]}\mathbb{R}[G]\otimes_{(\phi f,\gamma f,t)} C_*.$$

□

5 L^2-Alexander torsion for CW-complexes

5.1 Definition of the L^2-Alexander torsion for CW-complexes and manifolds

We now apply the developed tools to topology. Let π be a group and X be free π-CW-complex of finite type. Suppose $\phi\colon \pi \to \mathbb{R}$ and $\gamma\colon \pi \to G$ are compatible. We define a function

$$\begin{aligned} \tau^{(2)}(X,\phi,\gamma)\colon \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \\ t &\longmapsto \tau^{(2)}(C_*^{\phi,\gamma,t}(X)). \end{aligned}$$

In the case that γ is the identity map we write $\tau^{(2)}(X,\phi)$. This function is not well defined as we will see in the next lemma. Therefore we need a weaker notion of equality. We say two functions $f,g : \mathbb{R}^+ \to \mathbb{R}^+$ are equivalent, if there exists a $r \in \mathbb{R}$, such that $f(t) = t^r g(t)$. This defines an equivalence relation. We write $f \doteq g$, if f and g belong to the same equivalence class.

Lemma 5.1. *The equivalence class of a function $\tau^{(2)}(X,\phi,\gamma)$ is well defined i.e. independent of the choice of pushout diagrams.*

Proof. Different choices of pushout diagrams yield a change of basis by permutation and multiplication with trivial units. The permutation invariance has already been dealt with in Proposition 4.7. Multiplication with trivial units corresponds to a diagonal matrix A with entries $g_i \in G$. We get

$$\begin{aligned} \det_{\mathcal{N}(G)} \rho(A,\phi,\gamma,t) &= \det_{\mathcal{N}(G)} \begin{pmatrix} t^{\phi(g_1)}g_1 & 0 & \dots & 0 \\ 0 & t^{\phi(g_2)}g_2 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & t^{\phi(g_n)}g_n \end{pmatrix} \\ &= \prod_{i=1}^{n} t^{\phi(g_i)} = t^r. \end{aligned}$$

Now the lemma follows from the transformation formula. □

Theorem 5.2 (topological invariance)**.** *Let X and Y be free π-CW-complexes of finite type and $\phi\colon \pi \to \mathbb{R}$, $\gamma\colon G \to \mathbb{R}$ are compatible. Let $f\colon X \to Y$ be a cellular π-homeomorphism. Then we have*

$$\tau^{(2)}(X,\phi,\gamma,t) \doteq \tau^{(2)}(Y,\phi,\gamma,t)$$

Proof. Consider the short exact sequence of free $\mathbb{Z}[\pi]$-modules.

$$0 \longrightarrow C_*[1](X) \longrightarrow \operatorname{cone}_*(f) \longrightarrow C_*(Y) \longrightarrow 0$$

Here $C_*[1]$ denotes the suspension of C_* by 1 i.e.

$$C_n[1] := C_{n-1} \quad \text{and} \quad c_n[1] := (-1)^n c_{n-1},$$

Moreover

$$\operatorname{cone}_n(f) := C_n[1](X) \oplus C_n(Y) \quad \text{and} \quad \partial_n := \begin{pmatrix} c_n[1](X) & f_n \\ 0 & c_n(Y) \end{pmatrix}.$$

The basis of $C_*(X)$ and $C_*(Y)$ induce a basis for $\operatorname{cone}_*(f)$. Notice that the map i_n and p_n are than given by matrices with 1 on the main diagonal and 0 elsewhere. Therefore $\rho(i_n, \phi, \gamma, t)$ and $\rho(p_n, \phi, \gamma, t)$ satisfy the assumption of Lemma 4.4 and we have

$$\tau^{(2)}(C_*(X)[1]^{\phi,\gamma,t}) \cdot \tau^{(2)}(Y, \phi, \gamma, t) = \tau^{(2)}(\operatorname{cone}_*(f)^{\phi,\gamma,t}).$$

It follows directly from the definition that

$$\tau^{(2)}(C_*(X)[1]^{\phi,\gamma,t}) = \tau^{(2)}(X, \phi, \gamma, t)^{-1}.$$

Now combining the work of Whitehead about simple homotopy theory [Coh73, Theorem 22.2] and Chapman [Cha74, Theorem 1], we get $\tau^{(2)}(\operatorname{cone}_*(f)^{\phi,\gamma,t}) = t^r \doteq 1$.

□

Definition 5.3 (Admissible triple)**.** Let N be a connected finite CW-complex, $\phi \in H^1(N, \mathbb{R})$, and $\gamma\colon \pi_1(N) \to G$ a group homomorphism. When ϕ and γ are compatible, we call (N, ϕ, γ) an admissible triple.

Definition 5.4 (L^2-Alexander torsion for an admissible triple)**.** Let (N, ϕ, γ) be an admissible triple. The universal cover $\widetilde{N}$ is a $\pi_1(N)$-CW-complex and we define:

$$\tau^{(2)}(N, \phi, \gamma) := \tau^{(2)}(\widetilde{N}, \phi, \gamma).$$

This definition was first introduced by Li and Zhang in [LZ06]. In their paper they allow $t \in \mathbb{C}^*$, but show

$$\tau^{(2)}(N, \phi, \gamma)(t) = \tau^{(2)}(N, \phi, \gamma)(|t|).$$

That is why we have restricted to $t \in [0, \infty)$ from the beginning.

We will take the time to address some problems about the existence of the function $\tau^{(2)}(N, \phi, \gamma)$, even if they not appear in this thesis.

Recall the definition of a chain complex to be L^2-Det-acyclic. The chain complex must have trivial L^2-homology and every morphism has to be of determinant class. This converts to two natural question about L^2-Alexander torsion $\tau^{(2)}(N, \phi, \gamma)$ viewed as a function in $t \in \mathbb{R}^+$.

1. If $C_*(\widetilde{N})^{\phi,\gamma,t}$ is weakly acyclic for one $t \in \mathbb{R}^+$, will it be weakly acyclic for every $t \in \mathbb{R}^+$?
2. If $C_*(\widetilde{N})^{\phi,\gamma,t}$ has differentials of determinant class for one $t \in \mathbb{R}^+$, will it have differentials of determinant class for every $t \in \mathbb{R}^+$?

These two questions seems to have a positive answer by the unpublished work of Friedl and Lück [FL15],[Lüc15] and independently by the likewise unpublished work of [Liu15]. The relevance of these results will be discussed in the last Section of this thesis.

5.2 Basic properties of L^2-Alexander torsion

Some basic properties are stated here. Each is a direct consequence of Section 4.

Proposition 5.5. *Let (N, ϕ, γ) be an admissible triple and let $r \in \mathbb{R}$, then*

$$\tau^{(2)}(N, r\phi, \gamma)(t) = \tau^{(2)}(N, \phi, \gamma)(t^r).$$

Proof. We have already seen this kind of argument in Lemma 4.6. We take a generator $h \in \mathbb{Z}[\pi]$ and consider

$$\kappa(r\phi, \gamma, t)(h) = t^{r\phi(h)}\gamma(h) = \kappa(\phi, \gamma, t^r).$$

□

Another simple property following from Lemma 4.6 is

Proposition 5.6. *Let (N, ϕ, γ) be an admissible triple and let $\varphi : G \to H$ be a monomorphism. Then*

$$\tau^{(2)}(N, \phi, \varphi \circ \gamma) = \tau^{(2)}(N, \phi, \gamma).$$

Proposition 4.8 becomes:

Proposition 5.7. *Let (N, ϕ, γ) be an admissible triple. Let $p : \widehat{N} \to N$ be a finite regular cover such that $\ker(\gamma) \subset \widehat{\pi} := \pi_1(\widehat{N})$. We write $\widehat{\phi} = p^*\phi$ and we denote by $\widehat{\gamma}$ the restriction of γ to $\widehat{\pi}$. Then*

$$\tau^{(2)}(\widehat{N}, \widehat{\phi}, \widehat{\gamma}) \doteq \tau^{(2)}(N, \phi, \gamma)^{[\hat{N}:N]}.$$

Example 5.8. We look at the L^2-Alexander torsion $\tau^{(2)}(S^1, \phi, \gamma)$ of a circle with γ having infinity image. Since $\pi_1(S^1) = \mathbb{Z}$, γ has infinity image is equivalent with γ being injective. By Proposition 5.6 we can assume γ is the identity map. As in Example 2.13:

$$0 \longrightarrow l^2(\mathbb{Z}) \xrightarrow{g-1} l^2(\mathbb{Z}) \longrightarrow 0 \ .$$

Now applying the L^2-Alexander torsion to this chain complex yields:

$$\tau^{(2)}(S^1, \phi) = \det_{\mathcal{N}(G)}\left(t^{\phi(g)}g - 1\right) = \max\left\{1, t^{-\phi(g)}\right\}.$$

In view of Lemma 7.2 it might be useful, to rewrite the result. We can act on S^1 with S^1 via

$$\begin{aligned} S^1 \times S^1 &\longrightarrow S^1 \\ (t, z) &\longmapsto t^n z. \end{aligned}$$

The isotropy group H of one point v is the cyclic group $H = \langle e^{2\pi i/n}\rangle$ and we have $|H| = n$. The orbit of the action corresponds to the element $g^n \in \pi_1(S^1)$. We write $k_\phi = \phi(g^n) = n \cdot \phi(g)$ and get:

$$\tau^{(2)}(S^1, \phi) = \max\left\{1, t^{-\phi(g^n)/|H|}\right\} = \max\left\{1, t^{-k_\phi/|H|}\right\}.$$

5.3 Pullback of Alexander torsion

Let $f\colon Y \to X$ be a continuous cellular map between finite connected CW-complexes. We obtain two possible definitions for the pullback $f^*(\tau^{(2)}(X,\phi,\gamma))$ of an admissible triple (X,ϕ,γ). First by definition we easily see, that $(Y,\phi f_*,\gamma f_*)$ is an admissible triple, where $f_* : \pi(Y) \to \pi(X)$ is the induced map on the fundamental group. On the other site we can pullback the universal cover $f^*\widetilde{X}$. Then $C_* := C_*(f^*\widetilde{X})$ is a free $\mathbb{Z}[\pi(Y)]$-module and we can calculate the torsion $\tau^{(2)}(C_*,\phi,\gamma)$. The next lemma states that if one exists both coincide.

Lemma 5.9. *Let $f : Y \to X$ be a continuous cellular map between finite connected CW-complexes. Then*

$$\tau^{(2)}(Y,\phi f_*,\gamma f_*) \doteq \tau^{(2)}(C_*^{\phi,\gamma,t}(f^*\widetilde{X})).$$

Proof. After choosing a fundamental domain for the action of $\pi_1(Y)$ and $\pi_1(X)$ on $\widetilde{Y}$ and $\widetilde{X}$ respectively, we have a homeomorphism

$$\begin{aligned} f^*\widetilde{X} &= \left\{ (\widetilde{x},y) \in \widetilde{X} \times Y \mid p(\widetilde{x}) = f(y) \right\} \\ &\cong \pi_1(X) \times_{f_*} \widetilde{Y}, \end{aligned}$$

which translate to an isomorphism $C_*(f^*\widetilde{X}) \cong \mathbb{Z}[\pi_1(X)] \otimes_{f_*} C_*(\widetilde{Y})$. The Alexander torsion is a topological invariant, so that the torsion does not depend on the choice of fundamental domain. The claim follows from Proposition 4.10. □

Remark 5.10. In the previous lemma and in the following discussion it is not really necessary, that Y is connected. We could define the torsion of an non connected space as the product of the torsion of each component. For simplicity, we suppress the case of Y having more than one connected component.

Let $i\colon Y \hookrightarrow X$ be a subcomplex. As described before we can pullback the universal cover of $\widetilde{X}$. In this case we get an even simpler description. Let $p\colon \widetilde{X} \to X$ be the covering map, then $i^*\widetilde{X} \cong p^{-1}(Y)$. Therefore we can view $p^{-1}(Y)$ as a subcomplex of $\widetilde{X}$.

Definition 5.11 (L^2-Alexander torsion of pairs)**.** Let (X,ϕ,γ) be an admissible triple, $p\colon \widetilde{X} \to X$ the universal cover, and $Y \subset X$ a subcomplex. Denote $\widetilde{Y} := p^{-1}(Y)$. Then $C_*(\widetilde{X})/C_*(\widetilde{Y})$ is free $\pi_1(X)$-module and we define:

$$\tau^{(2)}(X,Y,\phi,\gamma,t) := \tau^{(2)}(C_*^{\phi,\gamma,t}(\widetilde{X},\widetilde{Y})).$$

Lemma 5.12. *Let (X,ϕ,γ) be an admissible triple, $p\colon \widetilde{X} \to X$ the universal cover, and $i\colon Y \hookrightarrow X$ a subcomplex. Denote $\pi := \pi_1(X)$ and $\widetilde{Y} = p^{-1}(Y)$. If two out of three of $\tau^{(2)}(X,\phi,\gamma,t), \tau^{(2)}(X,\phi i_*,\gamma i_*,t)$ and $\tau^{(2)}(X,Y,\phi,\gamma,t)$ exist, then although the third one exists and we have the identity:*

$$\tau^{(2)}(X,Y,\phi,\gamma,t) \doteq \frac{\tau^{(2)}(X,\phi,\gamma,t)}{\tau^{(2)}(Y,\phi i_*,\gamma i_*,t)}.$$

Proof. We get the short exact sequence of free $\mathbb{Z}[\pi]$-modules:

$$0 \longrightarrow C_*(\widetilde{Y}) \xrightarrow{i_*} C_*(\widetilde{X}) \xrightarrow{p_*} C_*(\widetilde{X}, \widetilde{Y}) \longrightarrow 0.$$

Since $C_*(\widetilde{Y})$ is a subcomplex of $C_*(\widetilde{X})$ the maps are given by matrices, which have 1 on the main diagonal and zero elsewhere. We obtain the short exact sequence:

$$0 \longrightarrow C_*^{\phi,\gamma,t}(\widetilde{Y}) \xrightarrow{i_*^{\phi,\gamma,t}} C_*^{\phi,\gamma,t}(\widetilde{X}) \xrightarrow{p_*^{\phi,\gamma,t}} C_*^{\phi,\gamma,t}(\widetilde{X}, \widetilde{Y}) \longrightarrow 0.$$

Still the maps $i_*^{\phi,\gamma,t}$ and $p_*^{\perp,\phi,\gamma,t}$ are isometrics and we can apply the product formula 4.4. We get:

$$\tau^{(2)}(C_*^{\phi,\gamma,t}(\widetilde{X}, \widetilde{Y})) \cdot \tau^{(2)}(C_*^{\phi,\gamma,t}(\widetilde{Y})) = \tau^{(2)}(C_*^{\phi,\gamma,t}(\widetilde{X})).$$

In Lemma 5.9 we have seen, that $\tau^{(2)}(C_*^{\phi,\gamma,t}(\widetilde{Y})) = \tau^{(2)}(Y, \phi i_*, \gamma i_*, t)$. $\square$

Proposition 5.13 (Glueing formula)**.** *Given a pushout diagram*

$$\begin{array}{ccc} T & \xhookrightarrow{\;i_1\;} & N_1 \\ \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle j_1} \\ N_2 & \xrightarrow{\;j_2\;} & M \end{array}$$

such that every map is cellular and i_1 an closed embedding. Let (M, ϕ, γ) be an admissible triple and the torsion $\tau^{(2)}(N_1, \phi j_1, \gamma j_1)$, $\tau^{(2)}(N_2, \phi j_2, \gamma j_2)$, and $\tau^{(2)}(T, \phi j_1 i_1, \gamma j_1 i_1)$ exist, then $\tau^{(2)}(M, \phi, \gamma)$ exists. More over,

$$\tau^{(2)}(N_1, \phi j_1, \gamma j_1, t) \cdot \tau^{(2)}(N_2, \phi j_2, \gamma j_2, t) \doteq \tau^{(2)}(T, \phi j_1 i_1, \gamma j_1 i_1, t) \cdot \tau^{(2)}(M, \phi, \gamma, t).$$

Proof. The pushout diagram

$$\begin{array}{ccc} T & \xhookrightarrow{\;i_1\;} & N_1 \\ \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle j_1} \\ N_2 & \xrightarrow{\;j_2\;} & M \end{array}$$

yields the π-equivariant pushout diagram

$$\begin{array}{ccc} i_1^* j_1^* \widetilde{M} & \xhookrightarrow{\;i_1\;} & j_1^* \widetilde{M} \\ \big\downarrow{\scriptstyle i_2} & & \big\downarrow{\scriptstyle j_1} \\ j_2^* \widetilde{M} & \xrightarrow{\;j_2\;} & \widetilde{M} \end{array} .$$

We get an isomorphism by Proposition 2.1

$$C_*(j_1^*\widetilde{M}, i_1^*j_1^*\widetilde{M}) \cong C_*(\widetilde{M}, j_2^*\widetilde{M}).$$

By the topological invariance of the L^2-Alexander torsion 5.2 we have:

$$\tau^{(2)}(N_1, T, \phi j_{1*}, \gamma j_{1*}, t) \doteq \tau^{(2)}(M, N_2, \phi, \gamma, t).$$

All the assumptions of Lemma 5.12 are satisfied and we can apply it to both sides.

□

Next we are going to introduce Seifert fiber spaces and the Thurston norm to show that the L^2-Alexander torsion detects the Thurston norm of Seifert fiber spaces.

6 Seifert fiber spaces

Seifert fiber spaces were first introduced by Seifert [Sei32] in the year 1932. Since then they play an important role in the study of 3-manifolds. The most mentionable applications are the JSJ-decomposition and the Thurston's geometrization conjecture. The theory of Seifert fiber spaces is well developed and there are a lot of great introduction to this subject for example [JN83] and [Sco83]. What we are interested in, is the Thurston norm (see Definition 6.16) of those spaces. All results about Seifert fiber spaces, which we need to prove Theorem 6.26 will be recalled.

Convention. In this master thesis a 3-manifold is an orientable, compact 3-dimensional manifold, with empty or toroidal boundary.

6.1 Definition

Definition 6.1 (fiber torus)**.** A fiber torus $V(p,q)$ is a solid torus with a fibration obtained from the trivial I-bundle over the disk D^2 by

$$I \times D^2/\sim \qquad (0,x) \sim (1, e^{2\pi i q/p}x),$$

where one fiber is of the form $\bigcup_{k=1}^{p} I \times (e^{2\pi i q k/p}x)$.

A more descriptive picture is: Taking a solid torus with standard fibration, cutting it along a meridian and twist it by the angle $2\pi p/q$ as illustrated in Figure 2.

Definition 6.2. A fiber preserving homeomorphism of a fiber torus is an automorphism of a solid torus which sends fiber to fiber.

One can check that up to fiber preserving homeomorphism the number p is unique, while q can be chosen to be $\pm q \mod p$.

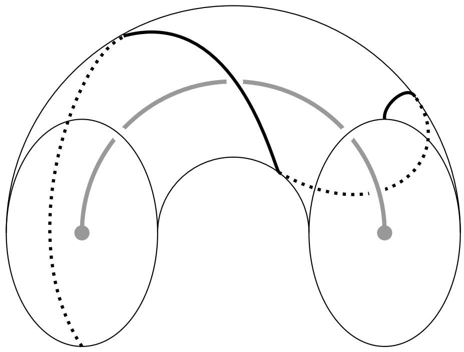


Figure 2: The middle fiber will be glued to itself, while the black fiber in the picture will be glued to a different fiber. The result is a fiber torus $V(2,3)$.

Definition 6.3. A Seifert fiber space M is a 3-manifold, which is the disjoint union of embedded circles, such that every circle has a tubular neighbourhood which is fiber preserving homeomorph to a fiber torus. A fiber is regular, if the fiber torus is $V(1,0)$ and singular or exceptional otherwise. The circles in the boundary give ∂M the structure of a trivial S^1-bundle over S^1.

Lemma 6.4. *The number of exceptional fibers is finite.*

Proof. By convention the manifold M is compact. All fibers in the neighbourhood of a singular fiber are regular. So the singular fibers lie discrete. Every infinite discrete subset of a compact space has an accumulation point. The neighbourhood of such an accumulation point is not a fiber torus and therefore the number of exceptional fibers must be finite. □

6.2 Examples of Seifert fiber spaces

In order to understand the definition better, we will go through a list of 3-manifolds which admit a Seifert fibration.

Example 6.5. This list will be discussed in more detail below.

- S^1-bundles
- 3-manifolds with fixed point free S^1 action
- A Seifert fiber space with a fiber torus removed
- Torus knots
- Lens spaces
- A finite covering of a Seifert fiber space

S^1-bundles are exactly the Seifert fiber spaces, which admit a Seifert fibration without a singular fiber. For example the 3-sphere is a S^1-principal bundle, via the Hopf fibration. This means the bundle structure comes from a free S^1-action. Therefore the 3-sphere is an example for the second item in the list, too. One can ask, if this is the only possible fibration of S^3 and the answer is no.

Lemma 6.6. *The 3-sphere admits different Seifert fibrations.*

Proof. We are thinking of the 3-sphere to be the set

$$S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$$

and fix some coprime numbers $p, q \in \mathbb{Z}$. Then there is an S^1-action:

$$\begin{aligned} S^1 \times S^3 &\longrightarrow S^3 \\ (\theta, z, w) &\longrightarrow (\theta^p z, \theta^q w). \end{aligned}$$

Now we want to see the Seifert fibration coming from this action. We can decompose S^3 in two solid tori $T_1 = \{(z, w) \in S^3 \mid |z|^2 \leq \frac{1}{2}\}$ and $T_2 = \{(z, w) \in S^3 \mid |w|^2 \leq \frac{1}{2}\}$ with $T_1 \cap T_2 \cong S^1 \times S^1$. The action restricts to T_1 and T_2 and it is not difficult to see that T_1 is fiber homeomorph to a fiber torus $V(p, q)$. Analogously $T_2 \cong V(q, p)$. □

Fixing the notation from the proof we see that one orbit in the intersection $T_1 \cap T_2$ is the standard embedding of a (p, q)-torus knot. It follows directly from the definition that a Seifert fiber space with a fiber torus removed is again a Seifert fiber space. So we deduce:

Lemma 6.7. *Let K be a (p, q)-torus knot and denote νK a tubular neighbourhood. Then $S^3 \setminus \nu K$ is a Seifert fiber space.*

Another example of Seifert fiber spaces are lens spaces. A lens space $L(a, b)$ is the quotient $S^3/\mathbb{Z}_a$ via the cyclic action

$$(z, w) \mapsto (e^{2\pi i/a} z, e^{2\pi i b/a} w).$$

This action commutes with the action above and so we get a Seifert fibration of a lens space $L(a, b)$. The fact that most Seifert fibrations come from S^1-actions is no coincidence as we will see in Section 6.5 below.

The last class of examples we discuss is the finite regular cover of a Seifert fiber space.

Lemma 6.8. *Let M be a 3-manifold with a given Seifert fibration and $p\colon \widehat{M} \to M$ a regular cover of degree d. Then p induces a Seifert fiber structure on $\widehat{M}$. Moreover let F be a regular fiber, k the number of connected components of $p^{-1}(F)$ and l the degree of the induced covering of one component $C \subset p^{-1}(F) \to F$. Then the equation $d = k \cdot l$ holds.*

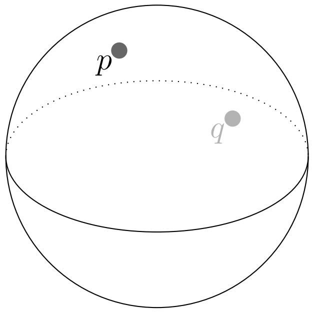


Figure 3: The base space of the Seifert fibration of S^3 defined in Lemma 6.6.

Proof. Let F be a fiber in M. It has the homeomorphism type of a circle, thus $p^{-1}(F)$ are disjoint circles and

$$\widehat{M} = \bigcup_{F \text{ is a fiber}} p^{-1}(F)$$

One has to check that the neighbourhood of a component $C \subset p^{-1}(F)$ is a (p, q)-fiber torus. This can be done by looking at the automorphisms of the solid torus. We skip the details.

Next we have to show, that the numbers l and k are independent of the choice of regular fiber and connected component. Assume F is a regular fiber and k the number of components of $p^{-1}(F)$. Since F is a regular fiber, there exists a neighbourhood U of F, such that every fiber F' in U is regular and $p^{-1}(F')$ has k components. So we see, having k components is an open condition. Every regular fiber can have at most d components covering it. Hence having not k components is an open condition, too. Both conditions cover M and since M is connected, we have that every fiber is covered by k connected components. Now let $C_1, C_2 \subset p^{-1}(F)$ be two components. Because the deck transformation group acts transitively, there is an element in the group of deck transformation, which induces an isomorphism between C_1 and C_2 as coverings. □

6.3 Invariants coming from a Seifert fibration

The basic way of thinking about a Seifert fiber space is as an S^1-bundle over an orbifold. We don't give a precise definition of this statement and refer to [Sco83]. The idea behind it is as follows. Every fiber has a neighbourhood which admits an S^1-action, via

$$\begin{aligned} S^1 \times S^1 \times D^2 &\longrightarrow S^1 \times D^2 \\ (z, s_1, r \cdot s_2) &\longmapsto (z^p s_1, r z^q s_2). \end{aligned}$$

The orbits of this action correspond to the fibers of the fiber torus. Now we can look at the quotient space. When we have a regular fiber torus, then the quotient space is D^2. When we have a (p, q)-fiber torus, then the quotient space is $D^2/\mathbb{Z}_p$, where $\mathbb{Z}_p$ acts by rotation $e^{2i\pi q/p}$. Topologically $D^2/\mathbb{Z}_p$ is just D^2. But we want to keep the additional information of this action. So the base space $\Sigma := M/S^1$ of a Seifert fiber space M is a pointed surface (see Figure 3). The points correspondence to the exceptional fibers and are decorated with the number p of the corresponding fiber torus $V(p,q)$.

Such spaces are intensively studied in complex geometry, where they appear as varieties of polynomials in two complex variables. The following invariant and the fact that it is multiplicative under finite cover comes from this area of mathematics.

Definition 6.9. (Orbifold Euler characteristic) Let Σ be a punctured surface, with points indexed by $p_1, \dots, p_n$. The orbifold Euler characteristic is defined by

$$\chi_{\text{orb}}(\Sigma) = \chi(\Sigma) - \sum_{i=1}^{n}(1 - \frac{1}{p_i}).$$

If there are no exceptional points we just have the ordinary Euler characteristic.

Example 6.10. Taking the surface of Figure 3 we get

$$\chi_{\text{orb}}(\Sigma) = \chi(S^2) - (1 - \frac{1}{p}) - (1 - \frac{1}{q}) = \frac{1}{p} + \frac{1}{q}.$$

Theorem 6.11. *(Riemann-Hurwitz formula) Let $f: \Sigma \to \Sigma'$ be a finite cover of orbifolds of degree k, then we have*

$$\chi_{orb}(\Sigma) = k \cdot \chi_{orb}(\Sigma').$$

Let $p\colon \widehat{M} \to M$ be a finite cover of Seifert fiber spaces. With the notation from Lemma 6.8 the factor k is the degree of the induced cover $p\colon \widehat{M}/S^1 \to M/S^1$. We want to extend the invariant $\chi_{\text{orb}}(M/S^1)$ to an invariant of Seifert fiber spaces, which is multiplicative under finite coverings. One could just correct the invariant by the factor l. However we go a step further and define a whole family of invariants. Taking an element $\phi \in H^1(M;\mathbb{R})$, we write

$$k_\phi(M) := \phi([F]),$$

where F is a regular fiber. Note that F has the homeomorphism type of a circle and therefore is an (possibly trivial) element in first homology. If M is clear from the context, we omit it from the notation.

Lemma 6.12. *Let M be a Seifert fiber space and $p\colon \widehat{M} \to M$ a finite covering. If $\widehat{M}$ is equipped with the Seifert fiber structure of Lemma 6.8 we have*

$$l \cdot k_\phi(M) = k_{p^*(\phi)}(\widehat{M})$$

for every $\phi \in H^1(M)$.

Proof. Let $\phi \in H^1(M)$ and F be a regular fiber. Let $C \subset p^{-1}(F)$ be a connected component. By definition we have

$$k_{p^*(\phi)}(\widehat{M}) = p^*\phi([C]) = \phi(p_*[C]) = \phi(l \cdot [F]) = l \cdot \phi([F]),$$

where we have used the fact $p_*[C] = l \cdot [F]$ and l is the degree of the covering $C \to F$. □

This lemma together with the Riemann-Hurwitz formula and Lemma 6.8 yields the following theorem:

Theorem 6.13. *Let M be a 3-manifold with a given Seifert fibration and $p : \widehat{M} \to M$ a regular covering of degree d. Let $\widehat{M}$ be given induced Seifert fiber structure. Then we have*

$$d \cdot k_\phi \cdot \chi_{orb}(\Sigma) = k_{p^*\phi} \cdot \chi_{orb}(\widehat{\Sigma}),$$

where Σ and $\widehat{\Sigma}$ are the base spaces of M and $\widehat{M}$.

6.4 Thurston norm of a Seifert fiber space

In this section we give a formula for the Thurston norm in terms of the invariants defined in the previous section. To begin we recall the definition and the basic properties of the Thurston norm.

Lemma 6.14. *Let M be a 3-manifold. Then the Poincaré dual of every element of $H^1(M, \mathbb{Z})$ can be represented by an orientable embedded surface.*

Definition 6.15 (Complexity of a surface)**.** Let S be a orientable surface and C_i its connected components. We define

$$\chi^-(S) := \sum_{C_i \neq S^2, D^2} -\chi(C_i)$$

as the complexity of S.

Definition 6.16 (Thurston norm)**.** Let M be a 3-manifold. We define

$$x_M(\phi) := \min \left\{ \chi^-(S_\phi) \ | S_\phi \text{ is dual to } \phi \right\}.$$

Theorem 6.17 ([Thu86])**.** *The map*

$$\begin{aligned} x_M \colon H^1(M; \mathbb{Z}) &\longrightarrow \mathbb{N} \\ \phi &\longmapsto x_M(\phi) \end{aligned}$$

extends to a seminorm on $H^1(M; \mathbb{R})$.

In the following homology and cohomology are with real coefficients if not stated otherwise.

6.4.1 The closed case

Lemma 6.18 (Thurston-norm of principal S^1-bundles)**.** *Let M be a closed prime 3-manifold with infinite fundamental group and $M \neq S^1 \times S^2$. Moreover let M be principal S^1-bundle over the base space Σ_g and $\phi \in H^1(M)$. Then the Thurston norm is given by*

$$x_M(\phi) = |\chi(\Sigma_g) \cdot k_\phi|$$

where k_ϕ is as defined before.

Before we proceed with the proof, we recall the definition of the transfer map.

Definition 6.19. (Freudenthal-Hopf) Let M be a closed orientable n-dimensional manifold. Let $PD_M : H^k(M) \to H_{n-k}(M)$ denote the Poincaré duality. For a map $f : M \to N$ to an m-dimensional closed manifold N, we define

$$f_! := PD_M f^* PD_N^{-1}$$

and

$$f^! := PD_N^{-1} f_* PD_M.$$

Proof. We split our investigation in two parts. First we prove the identity for the trivial bundle. By Künneth theorem we have:

$$H_2(M) = H_2(\Sigma_g) \oplus H_1(\Sigma_g) \otimes H_1(S^1).$$

Now let $\gamma \in H_1(\Sigma_g; \mathbb{Z})$ be a generator embedded in Σ_g. Then obviously $\gamma \times S^1$ is an embedded torus in M and a generator of $H_1(\Sigma_g) \otimes H_1(S^1)$. Since the Thurston norm is subadditive, elements with vanishing Thurston norm build a linear subspace. So we can ignore $H_1(\Sigma_g) \otimes H_1(S^1)$. It follows directly from the definition, that k_ϕ is linear in ϕ. Thus without loss of generality we restrict ourself to the case, where $\phi \in H^1(M)$ is dual to the generator $\Sigma_g \in H_2(\Sigma_g; \mathbb{Z})$. In this case we get $k_\phi = 1$. Let $i\colon S_\phi \hookrightarrow M$ represents Σ_g in $H_2(M)$, then it induces an isomorphism $(p \circ i)_* : H_2(S_\phi) \to H_2(\Sigma_g)$ and by Lemma 8.2 S_ϕ has at least the genus of Σ_g. Because $\Sigma_g \neq S^2$, we conclude

$$x_M(\phi) = -\chi(\Sigma_g) \cdot k_\phi.$$

Next we consider the case, where M is a non trivial S^1-bundle. In this case we will actually show, that the Thurston norm is trivial and the fiber is a trivial element of $H_1(M)$, and therefore k_ϕ is also zero. All this will be derived from the long exact sequence due to Gysin [Bre93, Theorem 13.2]:

$$0 \longrightarrow H^1(\Sigma_g) \xrightarrow{p^*} H^1(M) \xrightarrow{p^!} H^0(\Sigma_g) \xrightarrow{\cup e} H^2(\Sigma_g) \xrightarrow{p^*} H^2(M) \xrightarrow{p^!} H^1(\Sigma_g) \longrightarrow 0.$$

Here e is the Euler class. It is not trivial by the classification of S^1-bundle and the assumptions that S^1-bundle is non trivial. Therefore the map $\cup e$ is a isomorphism. Using the exactness of the sequence, we obtain, that $p^!$ is the zero map and thus p^* is surjective. But p^* is also injective and hence an isomorphism. To get a better understanding of this isomorphism, we change to homology using Poincaré duality.

$$0 \longrightarrow H_1(\Sigma_g) \xrightarrow{p_!} H_2(M) \xrightarrow{p_*} H_2(\Sigma_g) \xrightarrow{(\cup e)_!} H_0(\Sigma_g) \xrightarrow{p_!} H_1(M) \xrightarrow{p_*} H_1(\Sigma_g) \longrightarrow 0.$$

The map $p_!$ sends a cycle $c \in H_1(\Sigma_g)$ to its pre-image $p^{-1}(c)$ [GH14, Section 4] and in this case it is attaching a regular fiber. Thus every element in $H_2(M)$ is represented by tori and hence $x_M \equiv 0$. Now we show $k_\phi = 0$ by taking a second look to the Gysin sequence:

$$\dots \longrightarrow H_2(\Sigma_g) \xrightarrow{(\cup e)_!} H_0(\Sigma_g) \xrightarrow{p_!} H_1(M) \longrightarrow \dots$$

So $p_!$ is the zero map. The same argument as before $p_!$ sends a point to a regular fiber $[F] \in H_1(M)$ and therefore $[F] = 0$. Hence $k_\phi = \phi([F]) = 0$. □

The aim is to express the Thurston norm in terms of the invariants defined in the previous section. But these invariants depend on the choice of a Seifert fibration, which is not unique as we have seen for S^3. However the problem disappears when we restrict our attention to aspherical Seifert fiber spaces.

Theorem 6.20. *Let M be a Seifert fiber space and let $f\colon M \to N$ be a homeomorphism. Then f is homotopic to a fiber preserving homeomorphism unless one of the following occurs.*

1. *M is covered by S^3 or $S^1 \times S^2$.*
2. *M is covered by $S^1 \times S^1 \times S^1$.*

A proof of the theorem can be found in [Sco83, Theorem 3.9]. We will use the following conclusion of the theorem.

Corollary 6.21. *Let M be a closed S^1-bundle and aspherical. Then M admits a unique Seifert fiber structure up to fiber preserving homeomorphism.*

Proof. Since M is aspherical we only have to consider the case $M = S^1 \times S^1 \times S^1$. Given a Seifert fibration on $S^1 \times S^1 \times S^1$ with base orbifold Σ we have a short exact sequence [JN83, Corollary 6.3]:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \pi_1(M) \longrightarrow \pi_1^{\mathrm{orb}}(\Sigma) \longrightarrow 1$$

Here $\pi_1^{\mathrm{orb}}(\Sigma)$ is the orbifold fundamental group. It is defined by generators and relaters

$$\pi_1^{\mathrm{orb}}(\Sigma) := \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r \mid c_i^{p_i} = 1, \quad \prod [a_i, b_i] \cdot \prod c_i = 1 \rangle,$$

where a_i, b_i corresponds to underlying surface and c_i to the exceptional points with index p_i. In this case $\pi_1(M) = \mathbb{Z}^3$ and so $\pi_1^{\mathrm{orb}}(\Sigma)$ is an abelian group of rank two. Therefore we can change the presentation to:

$$\pi_1^{\mathrm{orb}}(\Sigma) := \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_r \mid c_i^{p_i} = 1, \quad \prod c_i = 1, \quad [a_i, b_i] = 1 \rangle,$$

One easily sees that, the only possibility to end up with an abelian group of rank two is $g = 1$ and all c_i are trivial. Which means we have a S^1-bundle over the torus. □

Theorem 6.22. *Let M be a aspherical closed Seifert fiber space. Fix a Seifert fiber structure on M and denote the base space Σ. Then the Thurston norm is given by*

$$x_M(\phi) = |\chi_{orb}(\Sigma) \cdot k_\phi|.$$

Proof. As shown in [AFW15, Section 3.2(C.10)] a Seifert fiber space is finitely covered by a S^1-principal bundle. Let $p : \widehat{M} \to M$ be such a cover of degree d. By Corollary 6.21 the Seifert fibration on $\widehat{M}$ is unique and therefore we may suppose $\widehat{M}$ carries the induced Seifert fiber structure. By the multiplicity of Theorem 6.13 we obtain:

$$k_\phi \, \chi_{\text{orb}}(\Sigma) = \frac{k_{p^*\phi} \, \chi_{\text{orb}}(\widehat{\Sigma})}{d}$$

By a famous theorem of Gabai [Gab83, Corollary 6.13], we have

$$x_M(\phi) = \frac{x_{\widehat{M}}(p^*\phi)}{d}.$$

and by Lemma 6.18 we conclude

$$|k_\phi \, \chi_{\text{orb}}(\Sigma)| = \frac{|k_{p^*\phi} \, \chi_{\text{orb}}(\widehat{\Sigma})|}{d} = \frac{x_{\widehat{M}}(p^*\phi)}{d} = x_M(\phi).$$

□

6.4.2 Doubling

The definition of Thurston norm still makes sense, when the manifold M has boundary. Then elements $S \in H_2(M, \partial M)$ are represented by properly embedded surfaces. This means, that the boundary of S is embedded in the boundary of M and only the boundary of S lies in the boundary of M. Theorem 6.17 still holds in this setting.

Example 6.23. In the special case of a knot exterior $M = S^3 \setminus \nu K$ it is $H^1(M) \cong \mathbb{R}$. The Thurston norm of a generator $\phi \in H^1(M; \mathbb{Z})$ fulfils the equation:

$$x_M(\phi) = 2 \cdot \text{genus}(K) - 1$$

For a (p, q)-torus knot K, it is well known that $\text{genus}(K) = \frac{(p-1)(q-1)}{2}$. In view of Theorem 6.26 we use the explicit description of the Seifert structure of a (p, q)-torus knot in Lemma 6.7 to regain this result. The base space Σ looks like Figure 3 with a disk removed, so that $\chi_{\text{orb}}(\Sigma) = \frac{1}{p} + \frac{1}{q} - 1 = \frac{p+q-pq}{pq}$. In addition we have $|k_\phi(S^3 \setminus \nu K)| = pq$, because the generator is the meridian $\mu \in H_1(S^3 \setminus \nu K)$ and in terms of this a regular fiber $[F] \in H_1(S^3 \setminus \nu K)$ is given by $[F] = pq \cdot \mu$. We obtain $x_M(\phi) = |\chi_{\text{orb}}(\Sigma) \cdot k_\phi| = |p+q-pq|$ and

$$\text{genus}(K) = \frac{x_M(\phi) + 1}{2} = \frac{1 - p - q + pq}{2} = \frac{(p-1)(q-1)}{2}.$$

To extend the results to manifolds with boundary we use a standard technique in low dimensional topology, which is called doubling. The procedure consists of the following steps:

1. Calculate the closed case.
2. Get a glueing formula along boundary.
3. Double the manifold by taking two disjoint copies and glue them together along their boundary.
4. Use step 1 and 2 to do the calculation.

In the previous section we have already dealt with part one. The glueing formula of the Thurston norm can be found in the book of Eisenbud and Neumann [EN85]:

Theorem 6.24. *Let T be a collection of incompressible disjoint tori embedded in N. Let B be a component of $N \setminus T$ and denote $i_B : B \to N$ the inclusion in N. For every $\phi \in H^1(M)$ the Thurston norm satisfies the equality*

$$x_N(\phi) = \sum_{B \subset N \setminus T} x_B(i_B^* \phi).$$

The glueing formula for the orbifold Euler characteristic is more familiar.

Lemma 6.25. *Let S be an orbifold and $U, V \subset S$ an open cover of S. We obtain*

$$\chi_{orb}(S) = \chi_{orb}(U) + \chi_{orb}(V) - \chi_{orb}(U \cap V)$$

This formula is proved in exactly the same way as for the regular Euler characteristic. If two orbifolds are glued together along their boundary, which is homeomorphic to S^1, then χ_{orb} becomes additive. We are ready to prove:

Theorem 6.26. *Let M be a prime orientable compact 3-manifold with infinite fundamental group, which is closed or has toroidal boundary. Let M be Seifert fibered with base space Σ_g and $M \neq S^1 \times S^2$, $S^1 \times D^2$. Then for any $\phi \in H^1(M)$, we have*

$$x_M(\phi) = |\chi_{orb}(\Sigma) \cdot k_\phi|,$$

where $k_\phi := \phi([F])$ and F is a regular fiber.

Proof. Most of the theorem has already been proven. We only need to consider the case, where M has boundary. Therefore we can double the manifold.

$$M \xhookrightarrow{i_1} M \textstyle\bigsqcup_{\partial M} M \xhookleftarrow{i_2} M,$$

which we denote M^{doub}. Then M^{doub} is not $S^1 \times S^2$, because $M \neq S^1 \times D^2$. Given a class $\phi \in H^1(M)$ we write $S_\phi \in H_2(M, \partial M)$ for a properly embedded dual i.e. $\partial S_\phi \subset$

∂M. The double S_ϕ^{doub} is a closed surface, which embeds in M^{doub} and thus $S_\phi^{\text{doub}} \in H_2(M^{\text{doub}})$. The Poincare dual ϕ^{doub} satisfies

$$i_1^*\phi^{\text{doub}} = i_2^*\phi^{\text{doub}} = \phi.$$

When we apply Theorem 6.24 to the tori of ∂M we get

$$x_{M^{\text{doub}}}(\phi^{\text{doub}}) = x_M(\phi) + x_M(\phi).$$

When Σ_g is the base space of M, then Σ_g^{doub} is the base space of M^{doub}. A regular fiber $F \in M \setminus \partial M$ can be seen as a regular fiber in M^{doub}. It is clear from the construction that $k_\phi(M) = k_{\phi^{\text{doub}}}(M^{\text{doub}})$. Hence,

$$\begin{aligned} k_{\phi^{\text{doub}}} \cdot \chi_{\text{orb}}(\Sigma_g^{\text{doub}}) &= k_\phi \left(\chi_{\text{orb}}(\Sigma_g) + \chi_{\text{orb}}(\Sigma_g)\right) \\ &= k_\phi \chi_{\text{orb}}(M) + k_\phi \chi_{\text{orb}}(M). \end{aligned}$$

Because M^{doub} is closed, we can apply Theorem 6.22 and we get:

$$x_M(\phi) = \frac{x_{M^{\text{doub}}}(\phi^{\text{doub}})}{2} = \frac{|k_{\phi^{\text{doub}}}|}{2} \cdot |\chi_{\text{orb}}(\Sigma_g^{\text{doub}})| = |k_\phi \chi_{\text{orb}}(\Sigma_g)|.$$

□

6.5 Seifert fiber spaces and S^1-actions

In this section we will discuss under which condition a Seifert fiber space M admits an S^1-action. We recall the well know slice theorem[Die87, Chapter 1, Theorem 5.6]:

Theorem 6.27 (Slice Theorem)**.** *Let M be a manifold on which a Lie group G acts effectively. For any $x \in M$ the map $G/G_x \to M$, $[g] \mapsto g \cdot x$ extends to an equivariant diffeomorphism in a neighbourhood. I.e for every $x \in M$ the orbit Gx has a neighbourhood, which is G-equivariant diffeomorph to $G \times_{G_x} T_xM/T_xGx$.*

This insures that a 3-manifold M, with a fixed point free S^1-action is Seifert fibered. In fact, in this case $G_x = \langle e^{2\pi i q/p}\rangle$ and $T_xM/T_xGx \cong D^2$. Hence the neighbourhood of an orbit is diffeomorph to $S^1 \times_{\langle e^{2\pi i q/p}\rangle} D^2$, which is exactly our model for a fiber torus $V(p,q)$. The opposite is true up to a cover of degree two as we will see in the next theorem.

Theorem 6.28. *Let M be a Seifert fiber space with orientable base space. Then M admits an S^1-action.*

Proof. We have described a local S^1-action before. For this action the number p is unique, while q leaves a choice of sign, which is choosing a direction for the action. So the local action can be extended to the whole manifold M, if and only if we can orient all fibers coherently, which is equivalent to have an oriented base space. □

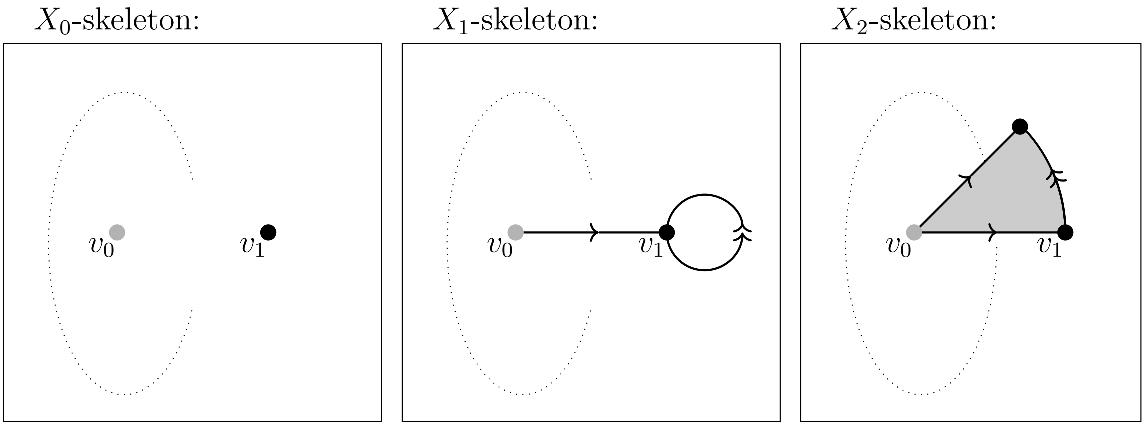


Figure 4: The filtration of the fiber torus $V(p,q)$. The dotted line indicates the S^1-orbit.

This action will be encoded in combinatorial data with the help of S^1-CW-complex, which has been defined in the Section 2.2. There is a general statement about G-equivariant triangulations [Ill83]. However in the simple case of a three manifold with an S^1 action we give a direct construction.

Theorem 6.29. *Every Seifert fiber space with orientable base space admits an S^1-CW-structure.*

Proof. We start with a S^1-CW-structure of a fiber torus $V(p,q)$. Morally one just gives a CW-structure to the space $V(p,q)/S^1$. This result is the following filtration: We have two 0-cells v_0 and v_1, two 1-cells e_0, e_1 and one 2-cell d as shown in Figure 4 and the characteristic maps are just the orbits.

Now let M be a Seifert fiber space with orientable base space. First we remove a neighbourhood of each exceptional fiber to end up with a trivial S^1-bundle over a surface Σ_g with toroidal boundary. For Σ_g we find a CW-structure and this gives a S^1-CW-structure for $S^1 \times \Sigma_g$. Now we can glue back the fiber tori, such that the fibers of the boundary torus coincide. Glueing is the same as a pushout diagram and since we have a S^1-CW-structure on the fiber torus, we get a S^1-CW-structure for the whole manifold M. □

7 L^2-Alexander torsion for Seifert fibered 3-manifolds

We have gathered everything to calculate the L^2-Alexander torsion for an interesting class of 3-manifolds, namely the Seifert fiber spaces. The goal of this section is the following theorem.

Theorem 7.1 (Main Theorem)**.** *Let M be a Seifert fiber space with $M \neq S^1 \times S^2, S^1 \times D^2$ and (M, ϕ, γ) an admissible triple, such that the image of a regular fiber under γ is an element with infinite order in G. The L^2-Alexander torsion is given by*

$$\tau^{(2)}(X, \phi, \gamma) = \max\left\{1, t^{x_M(\phi)}\right\}.$$

We will establish the theorem in two steps. First we show the theorem for M with orientable base space. Afterwards we use a glueing formula along tori to obtain the remaining cases.

7.1 L^2-Alexander torsion for S^1-CW-complex

Lemma 7.2 (L^2-torsion of S^1-CW-complex)**.** *Let X be a connected S^1-CW-complex of finite type and $\phi \in H^1(X;\mathbb{R})$. Suppose that for one and hence all $x \in X$ the map $ev_x\colon S^1 \to X$ defined by $z \mapsto z \cdot x$ induces an injective map $\gamma \circ ev_x\colon \pi_1(S^1, 1) \to G$. The composite*

$$\pi_1(S^1, 1) \xrightarrow{ev_x} \pi_1(X, x) \xrightarrow{\phi} \mathbb{R}$$

is given by multiplication with a real number. Let k_ϕ denote this number. Define the S^1-orbifold Euler characteristic of X by

$$\chi_{orb}^{S^1}(X) = \sum_{n \geq 0} (-1)^n \cdot \sum_{i \in J_n} \frac{1}{|H_i|}$$

where J_n denotes the set of open n-cells and for $i \in J_n$ the set H_i is the isotropy group of the corresponding cell. Then

$$\tau^2(X, \phi, \gamma) = \max\left\{1, t^{\chi_{orb}^{S^1}(X) \cdot k_\phi}\right\}$$

Proof. We use induction over the dimension of cells. In dimension zero X is a circle. There the theorem holds by example 5.8. Now the induction step from n-1 to n is done as follows. Per definition of S^1-CW-complex we can choose a equivariant S^1-pushout with $\dim(X_n) = n$

$$\begin{array}{ccc} \coprod_{i \in J_n} S/H_i \times S^{n-1} & \xrightarrow{\coprod q_i} & X_{n-1} \\ \downarrow i & & \downarrow j \\ \coprod_{i \in J_n} S/H_i \times D^n & \xrightarrow{\coprod Q_i} & X_n \end{array} .$$

And obtain from the glueing formula 5.13:

$$\prod_{i \in J_n} \tau^{(2)}(S^1/H_i \times D^n, S^1/H_i \times S^{n-1}, \gamma Q_i, \phi Q_i) = \tau^{(2)}(X_n, X_{n-1}, \gamma, \phi)$$

The left hand side can be computed by the suspension isomorphism 8.1.

$$\begin{aligned} \prod_{i \in J_n} \tau^{(2)}(S^1/H_i \times D^n, S^1/H_i \times S^{n-1}, \gamma Q_i, \phi Q_i) &= \prod_{i \in J_n} \tau^{(2)}(\widetilde{S^1}, \phi Q_i, \gamma Q_i)^{(-1)^n} \\ &= \prod_{i \in J_n} \max\left\{1, t^{(-1)^n k_\phi / |H_i|}\right\} \\ &= \max\left\{1, t^{(-1)^n k_\phi \sum_{i \in J_n} 1/|H_i|}\right\} \end{aligned}$$

Here we used the assumption that $\gamma \circ ev_x$ has infinite image. The torsion for X_{n-1} is defined by induction hypothesis. We can apply Lemma 5.12 and conclude:

$$\begin{aligned}\tau^{(2)}(X_n, \gamma, \phi) &= \tau^{(2)}(X_n, X_{n-1}\gamma, \phi) \cdot \tau^{(2)}(X_{n-1}, \gamma i, \phi i) \\ &= \max\left\{1, t^{(-1)^n k_\phi \sum_{i \in J_n} 1/|H_i|}\right\} \cdot \max\left\{1, t^{\chi^{S^1}_{\mathrm{orb}}(X_{n-1})k_\phi}\right\} \\ &= \max\left\{1, t^{\chi^{S^1}_{orb}(X) \cdot k_\phi}\right\}\end{aligned}$$

□

All Seifert fiber spaces with orientable base space admit a S^1-CW-structure by Theorem 6.29. One easily sees that $\chi_{\mathrm{orb}}(M/S^1) = \chi^{S^1}_{\mathrm{orb}}(M)$. With Theorem 6.26 we have verified Theorem 7.1 for all Seifert fiber spaces with orientable base space.

Example 7.3. Let K be a (p, q)-torus knot and $\phi \in H^1(S^3 \setminus \nu K; \mathbb{Z})$ a generator. We have already seen that $x_{S^3 \setminus \nu K}(\phi) = p + q - pq$. Therefore:

$$\tau^{(2)}(S^3 \setminus \nu K, \phi) = \max\left\{1, t^{p+q-pq}\right\}$$

We derive another useful fact from Lemma 7.2. Namely the vanishing of the L^2-Alexander torsion of a torus T^2. For the obvious S^1 action, we have

$$\chi^{S^1}_{\mathrm{orb}}(T^2) = \chi(S^1) = 0$$

Thus for any $\phi \in H^1(T^2)$ and compatible γ with infinite image i.e. $\tau^{(2)}(T^2, \phi, \gamma) \doteq 1$. Therefore the glueing formula 5.13 along tori reduces to:

Corollary 7.4. *Let (N, ϕ, γ) be an admissible triple. Assume N can be obtained by glueing 3-manifolds $N_1, \dots N_m$ along tori $T_1, \dots, T_n$ in their boundary such that the restriction of γ to each torus T_i has infinite image and the torsion $\tau^{(2)}(N_k, \phi i_k, \gamma i_k)$ exists for all $k \in \{1, \dots, m\}$. Then*

$$\prod_{k=1}^{m} \tau^{(2)}(N_k, \phi i_{k*}, \gamma i_{k*}) \doteq \tau^{(2)}(N, \phi, \gamma)$$

7.2 L^2-Alexander torsion for Seifert fiber spaces with non orientable base space

One should observe, that cutting a Seifert fiber space M along a torus T, such that the standard fibration of the torus and the fibration of M coincide is exactly the same as cutting the base space of M along S^1. This will be the key observation in both proofs.

Lemma 7.5. *Let M be a Seifert fiber space with a Klein bottle as base space and (M, ϕ, γ) admissible, such that the image of γ restricted to a fiber has infinite image. Then*

$$\tau^{(2)}(M, \phi, \gamma, t) = \max\left\{1, t^{x_M(\phi)}\right\}$$

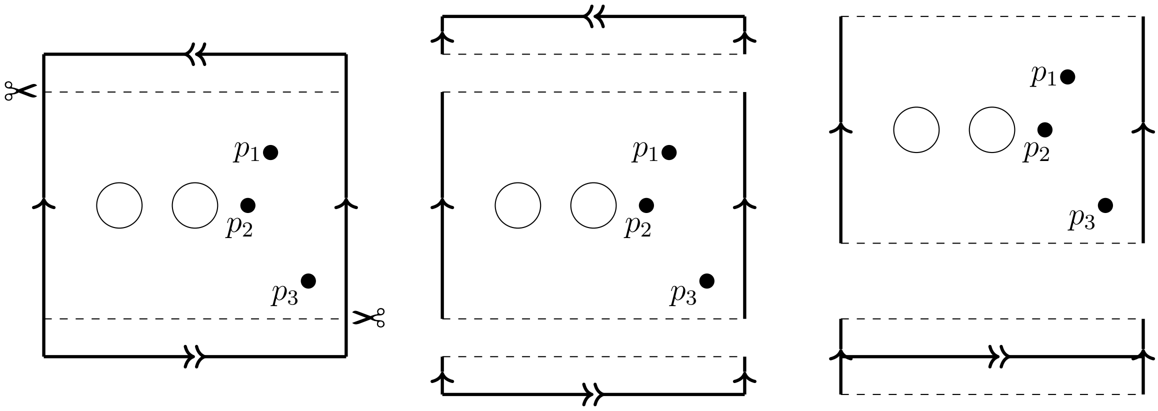


Figure 5: A Klein bottle with two boundary components and three exceptional points is cutted along two S^1 to obtain two orientable pieces.

Proof. We can cut M along two tori, such that we obtain two Seifert fibered pieces M_1, M_2 both with orientable base space (see Figure 5) and a fiber of M and the fibers of the two tori coincide. Then the restriction of γ to the tori has infinite image by assumption. We use the glueing formula along tori 7.4 and obtain

$$\begin{aligned}\tau^{(2)}(M,\phi,\gamma,t) &= \tau^{(2)}(M_1,\phi i_1,\gamma i_1,t)\cdot\tau^{(2)}(M_2,\phi i_2,\gamma i_2,t)\\ &= \max\left\{1,t^{x_{M_1}(i_1^*\phi)}\right\}\cdot\max\left\{1,t^{x_{M_2}(i_2^*\phi)}\right\}\\ &= \max\left\{1,t^{x_{M_1}(i_1^*\phi)+x_{M_2}(i_2^*\phi)}\right\}\\ &= \max\left\{1,t^{x_M(\phi)}\right\}\end{aligned}$$

□

Theorem 7.6. *Let M be a Seifert fiber space with a non orientable base space other than $\mathbb{R}P^2$ and (M,ϕ,γ) admissible, such that the image of γ restricted to a fiber has infinite image. Then*

$$\tau^{(2)}(M,\phi,\gamma,t) = \max\left\{1,t^{x_M(\phi)}\right\}$$

Proof. By the classification of non orientable surfaces, we can cut the base space along S^1, such that every piece is a Klein bottle with boundary or a Möbius stripe. This corresponds to cutting M along tori, such that every connected component has a Möbius stripe or Klein bottle as base space. Because of the additivity of the Thurston norm and the glueing formula along tori it is sufficient to proof the statement for every piece. The case of the Klein bottle has been dealt with in the lemma before. Since the doubling of a Möbius stripe is the Klein bottle, we can use exactly the same doubling argument as in Theorem 6.26 to receive the desired result. □

7.3 Outlook

In [DFL14] the authors introduced the concept of a function to be monomial in the limit and the degree of a function, which is monomial in the limit. We recall the definitions.

Definition 7.7 (monomial in the limit)**.** We say a function f is monomial in the limit, if there are $D, d \in R$ and non-zero real numbers $c, C \in \mathbb{R}$, such that

$$\lim_{t\to 0} \frac{f(t)}{t^d} = c \quad \text{and} \quad \lim_{t\to\infty} \frac{f(t)}{t^D} = C$$

Definition 7.8 (degree of a map)**.** If f is monomial in the limit, we define

$$\deg f := D - d$$

as the degree of f.

Suppose $f \doteq g$. We have that f is monomial in the limit, if and only if g is. In this case we have $\deg f = \deg g$. So the degree is an invariant of the weaker notion of equality of function. The main theorem 7.1 can be restated to

$$\deg \tau^{(2)}(N, \phi) = x_N(\phi)$$

for all Seifert fiber spaces N with infinite fundamental group and $N \neq S^1 \times S^2, S^1 \times D^2$. We have mentioned earlier one of the important results in 3-dimensional topology. It is the JSJ-decomposition and Thurston's geometrization conjecture:

Theorem 7.9. *[AFW15, Geometrization Theorem 1.7.6] Let N be a compact, orientable, irreducible 3-manifold with empty or toroidal boundary. There exists a (possibly empty) collection of disjointly embedded incompressible tori $T_1, \dots, T_m$ in N such that each component of N cut along $T_1 \cup \dots \cup T_m$ is hyperbolic or Seifert fibered. Furthermore any such collection of tori with a minimal number of components is unique up to isotopy.*

From this we obtain the following corollary.

Corollary 7.10. *Let N be a irreducible 3-manifold, such that every connected component of the JSJ-decomposition is a Seifert fiber space. Let (N, ϕ, γ) be admissible and γ has infinite image restricted to each fiber of each component. Then:*

$$\deg \tau^{(2)}(N, \phi, \gamma) = x_M(\phi)$$

Proof. By Theorem 7.1 we have for every component N_k of $N \setminus \bigcup_{i=1}^m T_m$:

$$\deg \tau^{(2)}(N_k, \phi i_{k*}, \gamma i_{k*}) = x_{N_k}(\phi)$$

Now the glueing formula along tori states

$$\begin{aligned}\deg \tau^{(2)}(N, \phi, \gamma) &= \sum_{k=1}^m \deg \tau^{(2)}(N_i, \phi i_{k*}, \gamma i_{k*}) \\ &= \sum_{k=1}^m x_{N_i}(i_k^* \phi) \\ &= x_N(\phi)\end{aligned}$$

□

The last thing to achieve a complete picture is to understand the L^2-Alexander torsion for a hyperbolic 3-manifold. Last month on the ArXiv appeared a paper by Friedl and Lück [FL15] and independently a paper by Liu [Liu15]. Both state the theorem:

Theorem 7.11. *Let M be a irreducible 3-manifold with infinite fundamental group and empty or toroidal boundary. Then we get for any element $\phi \in H^1(M;\mathbb{Z})$.*

$$\deg \tau^{(2)}(M,\phi) = x_M(\phi)$$

8 Appendix

Here we present two lemmas, which might be, but don't have to be subject of an algebraic topology course.

Lemma 8.1 (Suspension)**.** *Let X be a connected CW-complex and $\phi \in H^1(X,\mathbb{R})$ and $\gamma : \pi_1(X) \to \Gamma$ such that ϕ, γ are compatible. Then we have:*

$$\tau^{(2)}(X \times D^n, X \times S^{n-1}, \gamma, \phi, t) = \tau^{(2)}(X, \gamma, \phi, t)^{(-1)^n}$$

Proof. The universal cover of $X \times D^n$ is $\widetilde{X} \times D^n$, where $\widetilde{X}$ is the universal cover of X. We have

$$\begin{aligned} C_*(\widetilde{X} \times D^n, \widetilde{X} \times S^{n-1}) &:= C_*(\widetilde{X} \times D^n)/C_*(\widetilde{X} \times S^{n-1}) \\ &\cong C_*(\widetilde{X}) \otimes_{\mathbb{Z}} C_*(D^n)/C_*(\widetilde{X}) \otimes_{\mathbb{Z}} C_*(S^{n-1}) \\ &\cong C_*(\widetilde{X}) \otimes_{\mathbb{Z}} C_*(D^n, S^{n-1}) \end{aligned}$$

All the isomorphisms are well known and come from the CW-structure of the cartesian product of two CW-complexes. Therefore they don't change the torsion. So we have to understand the complex $C_*(D^n, S^{n-1})$. For the obvious CW-structure of D^n, with one 0-cell, one n-1-cell and one n-cell, we have S^{n-1} as a subcomplex with one 0-cell and one n-1-cell. Hence

$$C_i(D^n, S^{n-1}) = \begin{cases} \mathbb{Z} & i = n \\ 0 & \text{else} \end{cases}$$

By the definition of tensor products of chain complexes, we have

$$C_*(\widetilde{X} \times D^n, \widetilde{X} \times S^{n-1}) \cong C_*(\widetilde{X}) \otimes_Z C_*(D^n, S^{n-1}) = C_{*-n}(\widetilde{X})$$

□

Lemma 8.2. *Let Σ_g, Σ_{g+k} be surfaces with* genus$(\Sigma_j) = j$ *and $k > 0$. There cannot exist a continuous map $f : \Sigma_g \to \Sigma_{g+k}$, such that $f_* : H_2(\Sigma_g) \to H_2(\Sigma_{g+k})$ induces an isomorphism.*

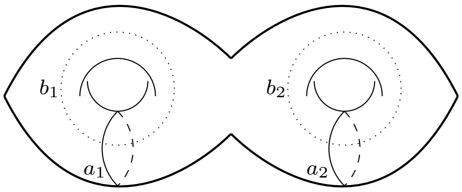


Figure 6: A surface of genus two, with the standard basis of $H^1(\Sigma_2)$.

Proof. There are isomorphisms $H^2(\Sigma_g) \cong \mathrm{Hom}\,(H_2(\Sigma_g), \mathbb{Z}) \cong \mathbb{Z}$. For an arbitrary $\phi \in \mathrm{Hom}\,(H_2(\Sigma_g), \mathbb{Z})$ we have $f^*(\phi) := \phi \circ f_*$. Thus f^* is an isomorphism if and only if f_* is an isomorphism. We show the statement for cohomology. Suppose there is a continuous map $f : \Sigma_g \to \Sigma_{g+k}$ such that $f^* : H^2(\Sigma_{g+k}) \to H^2(\Sigma_g)$ is an isomorphism. We use the fact that f^* is a ring homomorphism with respect to the cup product. Using the usual representation of a surface (see Figure 6), with the standard basis $\{a_1, b_1, \ldots, a_g, b_g\}$ for $H^1(\Sigma_g)$, we get a linear pairing by

$$\begin{aligned} H^1(\Sigma_g) \times H^1(\Sigma_g) &\longrightarrow H^2(\Sigma_g) \cong \mathbb{Z} \\ (\gamma_1, \gamma_2) &\longmapsto \gamma_1 \cup \gamma_2 \end{aligned}$$

satisfying $a_i \cup b_j = \delta_{ij}$ and $a_i \cup a_j = b_i \cup b_j = 0$.

Now the map $f^* : H^1(\Sigma_{g+k}) \to H^1(\Sigma_g)$ has at least a kernel of rank $2 \cdot k$. Hence there is an element $\gamma \in \ker f^* \setminus \{0\}$ with $\gamma = \sum_{i=1}^{g+k} \lambda_i a_i + \mu_i b_i$ and $\lambda_i \neq 0$ or $\mu_i \neq 0$ for some $i \in \{1, \ldots, g+k\}$. Without loss of generality we can suppose that $\lambda_i \neq 0$. Therefore we conclude

$$0 = f^*(\gamma) \cup f^*(b_i) = f^*(\gamma \cup b_i) = f^*(\lambda_i[\Sigma_{g+k}]) = \pm\lambda_i \cdot [\Sigma_g],$$

which is a contradiction. □

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